Cell Problems for Elastic Microstructures

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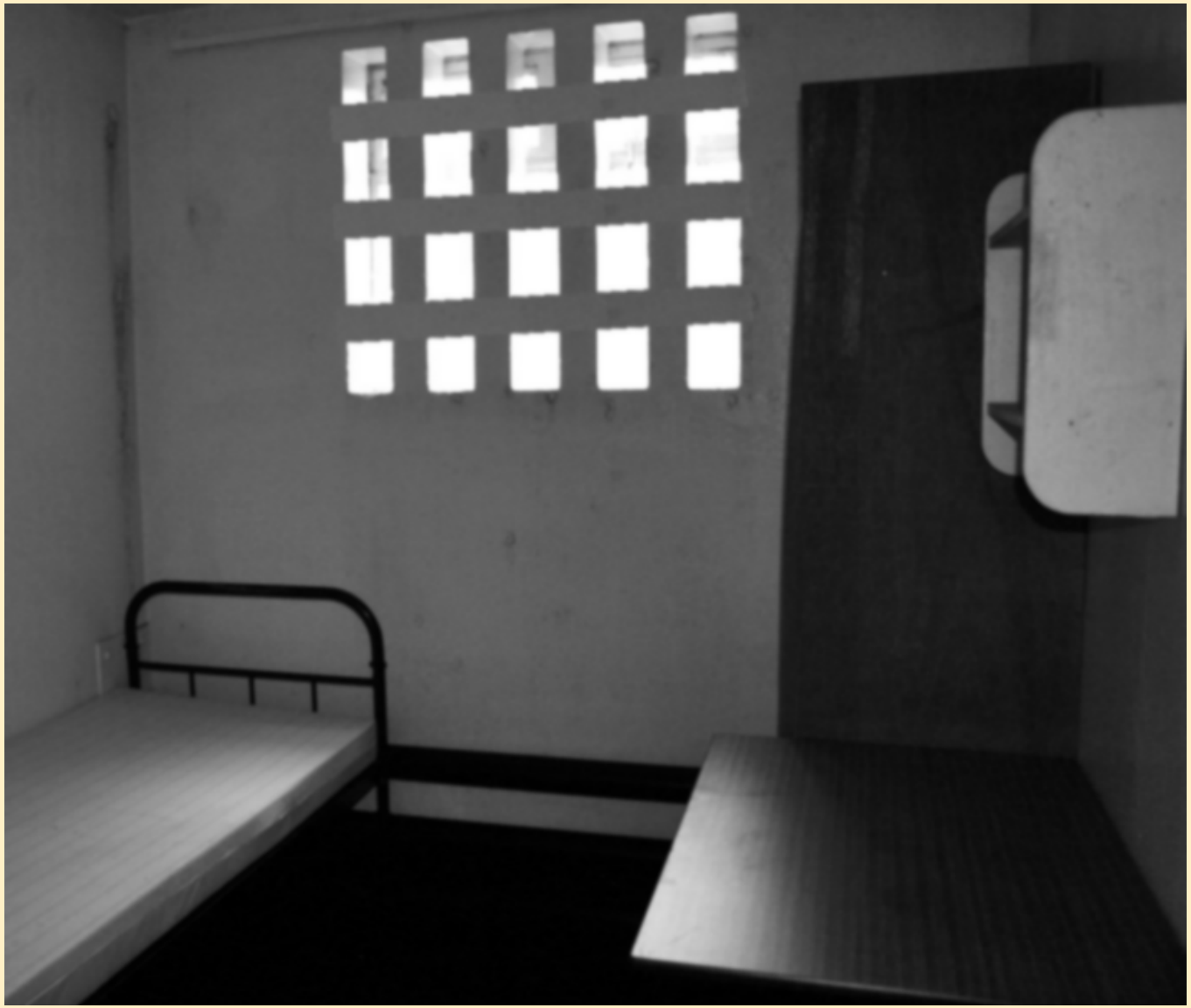
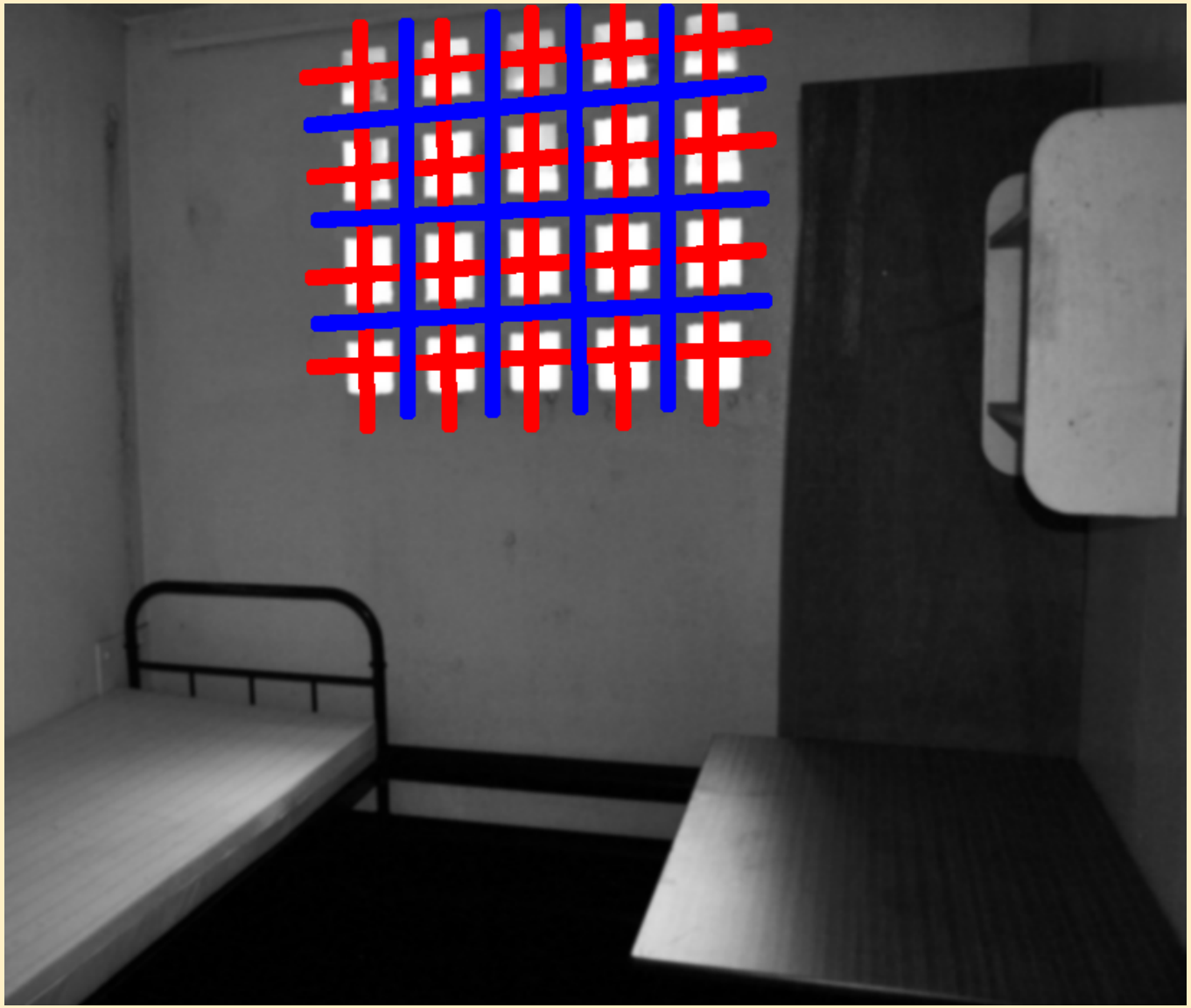
Oberseminar, Bonn
2008-05-08

Cell Problems?



adapted from an image by “Still Burning” on flickr.com

Multigrid and Cell Problems?



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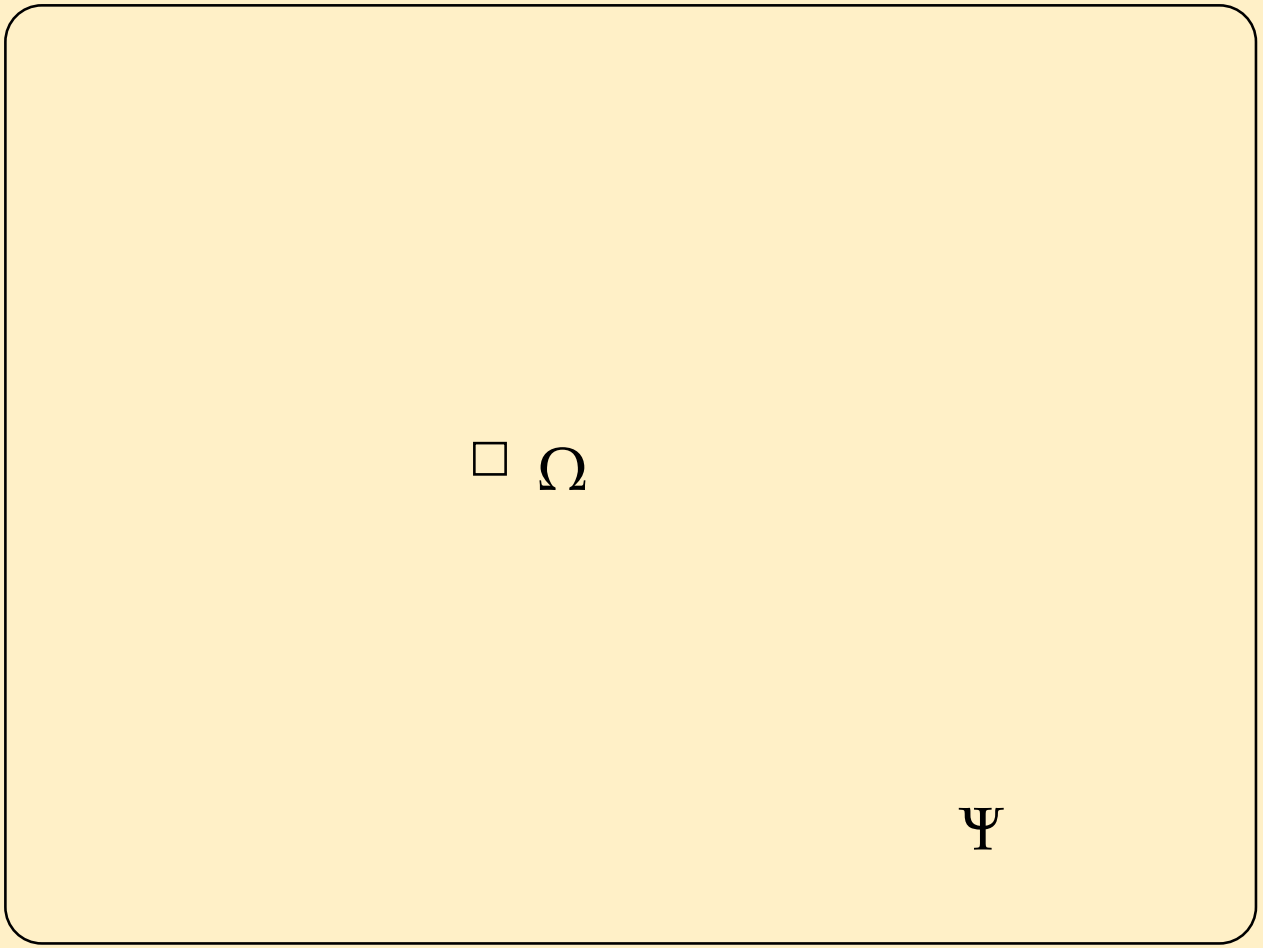
Outline

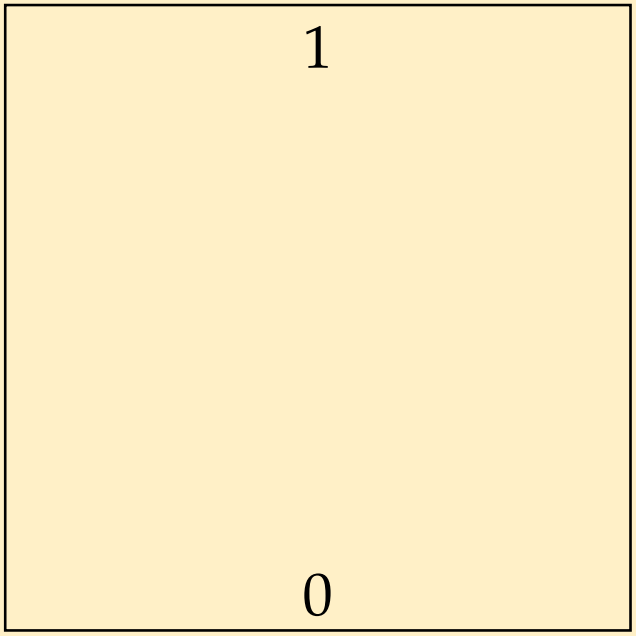
Contents

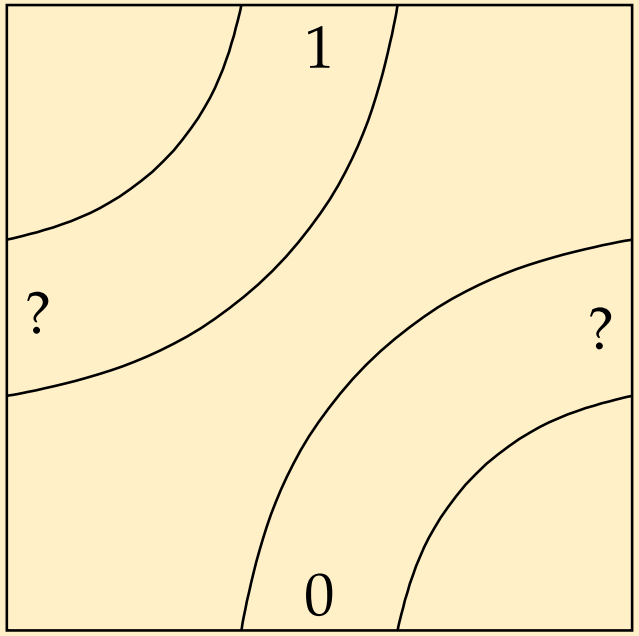
Homogenization



One periodic cell in a macroscopically homogeneous material.

Homogenization





Periodic cell can have internal microstructure.

Cell Problems

[Allaire 2002]

Idea: split u in

$$u = \bar{u} + \tilde{u}$$

where

$\bar{u}$ smooth macroscopic part on Ψ

$\tilde{u}$ oscillatory periodic part on Ω

Contents

Steady State of Heat Conduction

$$
\begin{aligned}
-\operatorname{div}\left(A(x)\nabla u(x)\right) = f(x) \quad & \text{in } \Psi \\
A : \Psi \to \mathbb{R}^{3\times 3} \quad & \Omega\text{-periodic} \\
f : \Psi \to \mathbb{R} \quad & \Omega\text{-periodic and } \fint_\Omega f = 0
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Split $u = \bar{u} + \tilde{u}$:

$$
\begin{aligned}
-\operatorname{div}(A(x)\nabla \tilde{u}(x)) &= f(x) + \operatorname{div}(A(x)\nabla \bar{u}(x)) \qquad \text{in } \Omega \\
\tilde{u} \quad & \text{periodic on } \Omega \qquad\qquad\qquad\qquad \text{(PS)} \\
\fint_\Omega \tilde{u} &= 0
\end{aligned}
$$

so that last two conditions make $\tilde{u}$ unique.

Macroscopic Heat Conduction Coefficient

Consider heat flux $Q = A\nabla u$ macroscopically on Ω:

$$\bar{Q}^{\Omega} = \bar{A}^{\Omega}\nabla\bar{u} \qquad (\spadesuit)$$

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$$\begin{aligned}\bar{Q}_i^{\Omega} &= \fint_{\Omega} Q_i(x) \\ &= \fint_{\Omega} A(x)\nabla u_i(x) \\ &= \fint_{\Omega} A(x)\left(\nabla\bar{u}_i(x) + \nabla\tilde{u}_i(x)\right)\end{aligned}$$

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Where

$$\mathcal{E}(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$$

is the “strain”.

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-\operatorname{div}\left(A(x)\mathcal{E}u(x)\right) = f(x) \quad & \text{in } \Psi \\
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\tilde{u} \quad &\text{periodic on } \Omega \qquad\qquad\qquad\qquad\qquad\qquad \text{(PV)} \\
\fint_\Omega \tilde{u} &= 0
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so that last 2 conditions make $\tilde{u}$ unique.

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so that last 6 conditions make $\tilde{u}$ unique.

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Note that

$$\mathcal{E}(u) = 0 \quad \forall u : u(x) = Sx + b$$

with S skew-symmetric 3×3 matrix and $b \in \mathbb{R}^3$.

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Note that

$$\mathcal{E}(u) = 0 \quad \forall u : u(x) = Sx + b$$

with S skew-symmetric 3×3 matrix and $b \in \mathbb{R}^3$.
BUT: Periodicity does not allow nonzero skew-symmetric part in $\tilde{u}$.

Elasticity

Consider stress $\sigma = A\mathcal{E}(u) = A\epsilon$ macroscopically on Ω:

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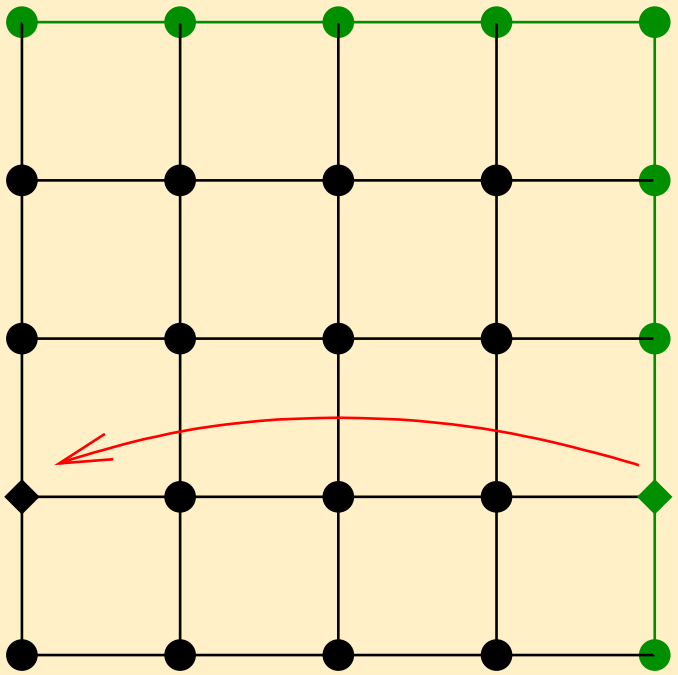
$$\begin{aligned}
\bar{\sigma}^{\Omega}_{i,j} &= \fint_{\Omega} \sigma_{i,j}(x) \\
&= \fint_{\Omega} A(x)\mathcal{E}(u_{i,j})(x) \\
&= \fint_{\Omega} A(x)\left(\mathcal{E}(\bar{u}_{i,j})(x) + \mathcal{E}(\tilde{u}_{i,j})(x)\right)
\end{aligned}$$

Contents

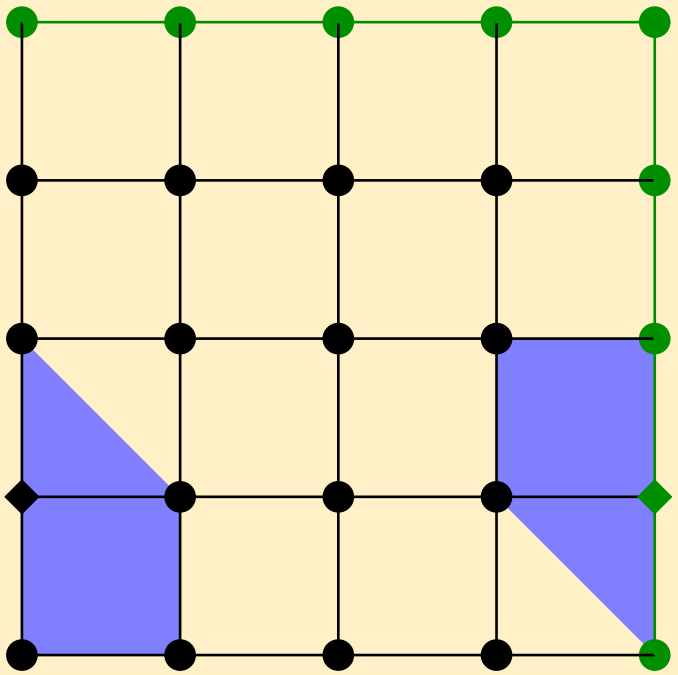
Contents

Periodic Boundary Conditions



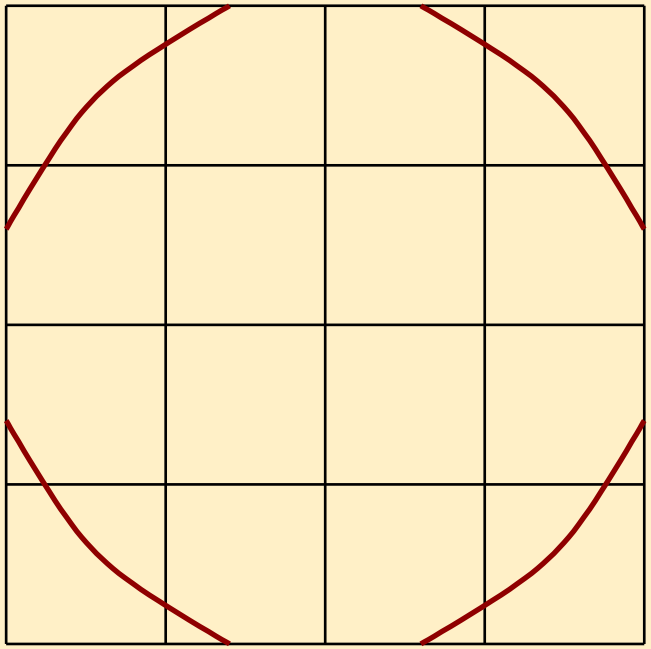
Data at *periodic nodes* is the same as at *facing present nodes*.

Basis functions



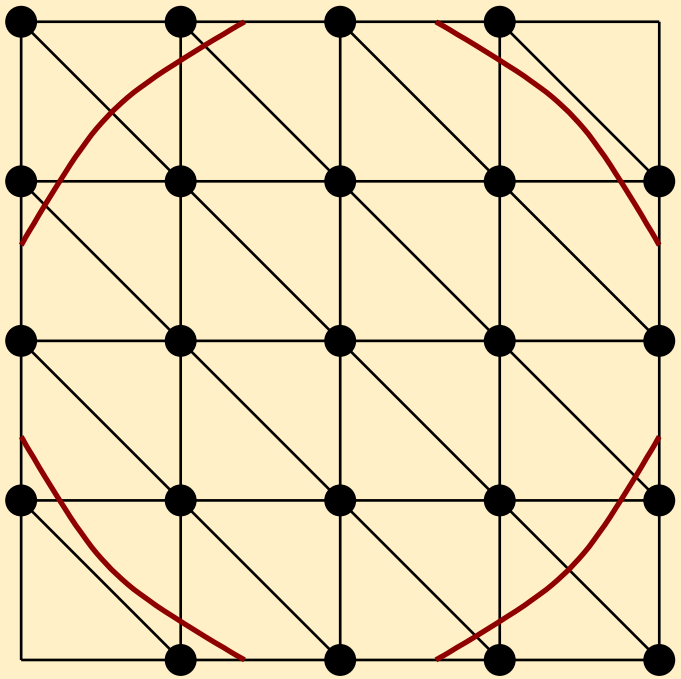
Identifying nodes (DOFs) also means identifying (M)FE basis functions, thus supports are modified.

Periodic Nodes for CFE



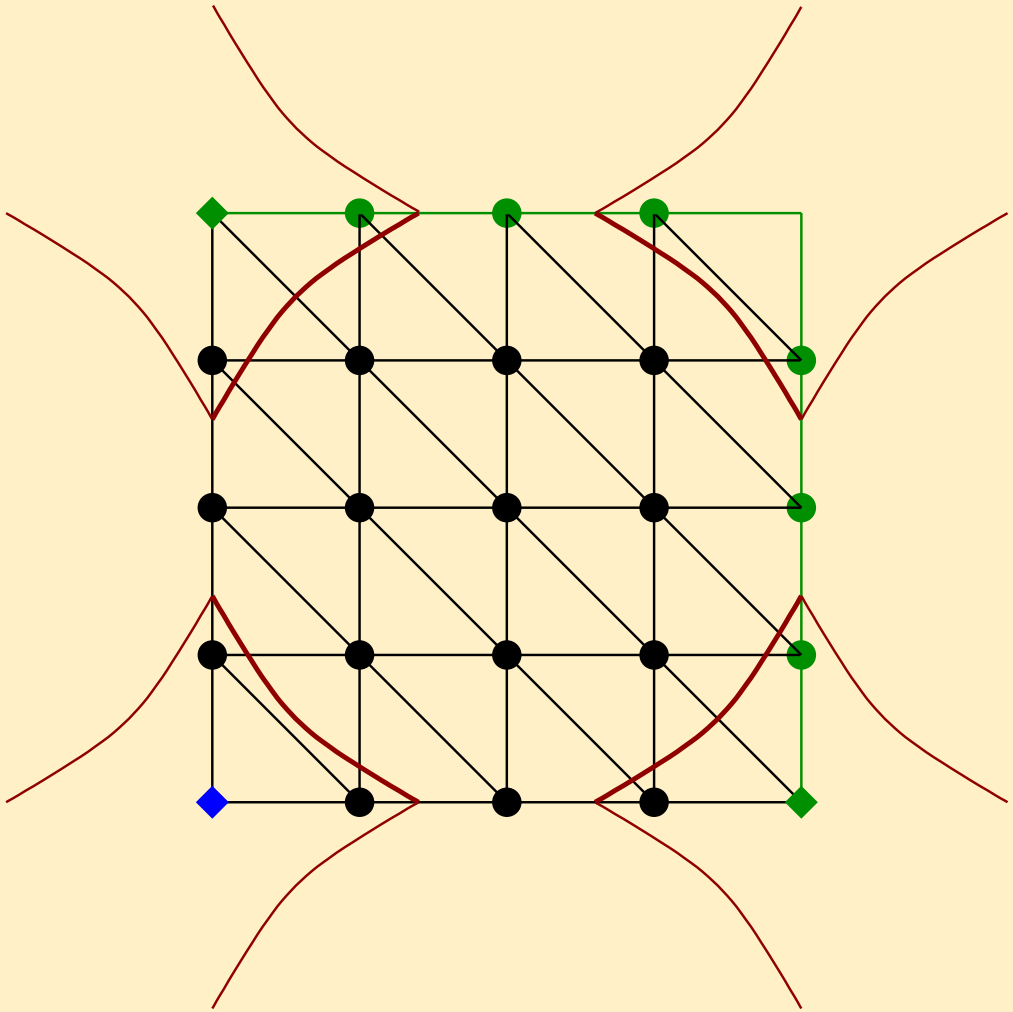
Geometric situation at top and bottom is obviously symmetric ...

Periodic Nodes for CFE



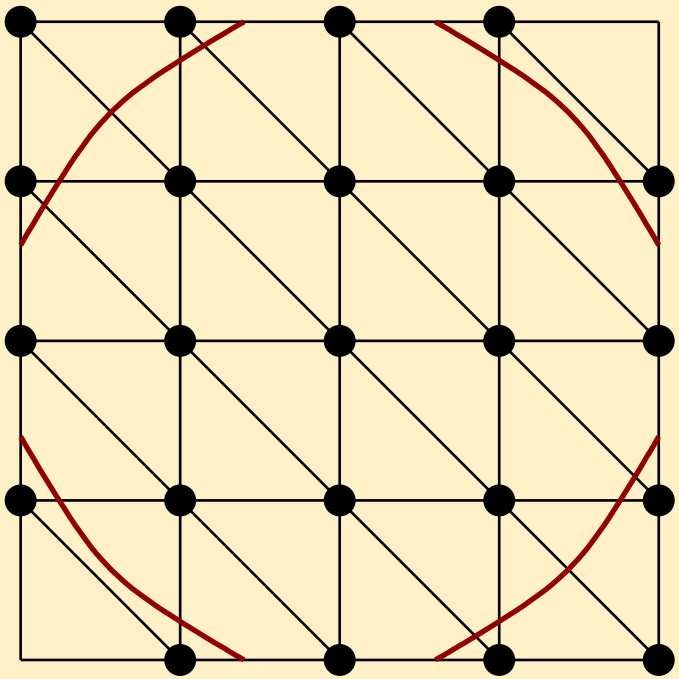
scheme of DOFs in a CFE grid at top and bottom is not symmetric!

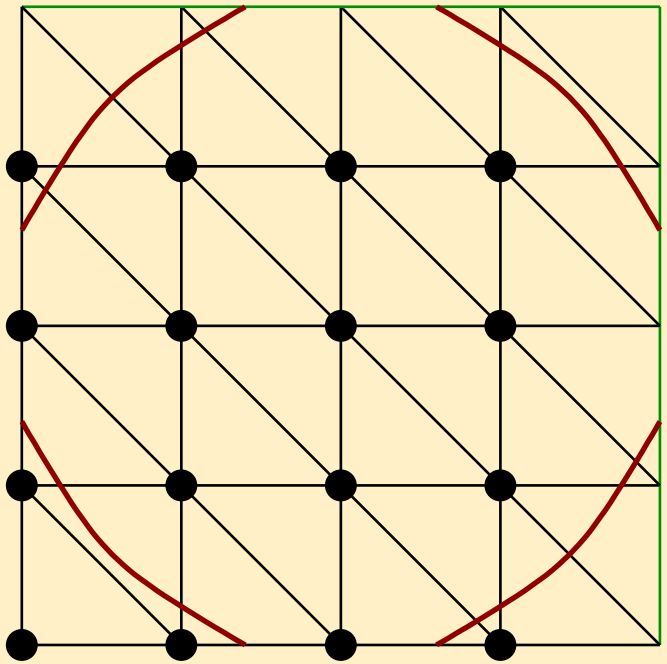
Periodic Nodes for CFE



bottom left node is facing present node of periodic DOFs—but not "originally" a DOF

Periodic Nodes for CFE





set of DOFs in a CFE grid and set of *present DOF nodes* for the periodic problem

Periodicity Handling

class `PeriodicityHandler`: general framework for periodicity, providing

- node periodic?
- facing present node
- matrix and vector modification

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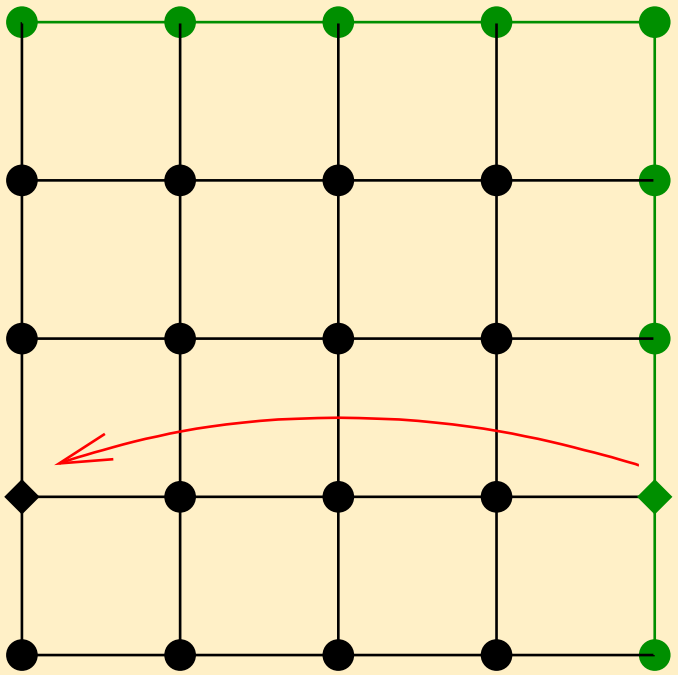
...soon to be part of the quocmesh library.

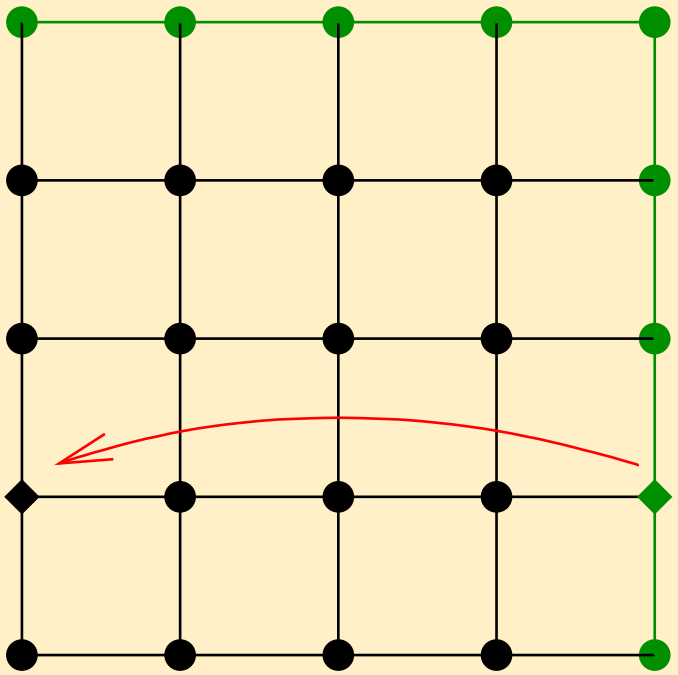
Contents

Periodic Restriction ($\sharp$)



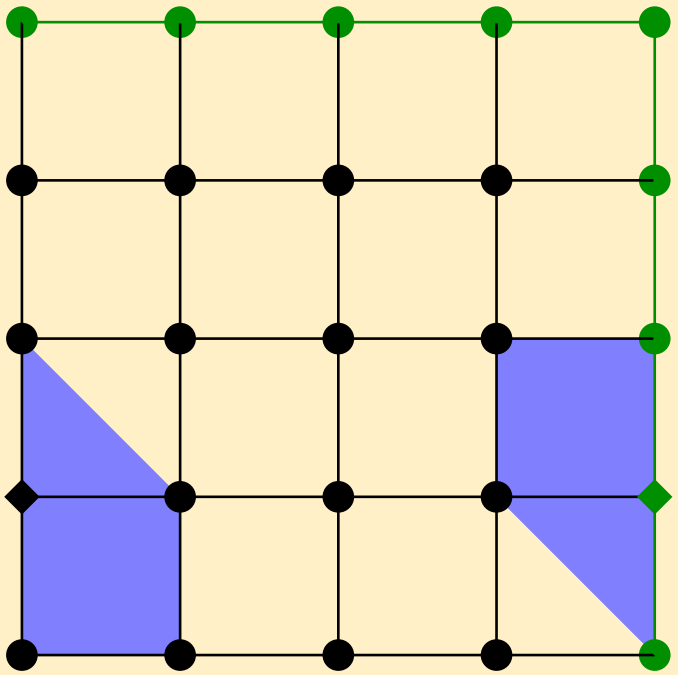
do not store data for periodic nodes.

Periodic Restriction ($\sharp$)



do not store data for periodic nodes.
$\sharp^{-1}$: extension: copy values from facing present nodes to periodic nodes.

Collapsing ($\natural$)



vectors with integrated quantities and matrices:
union of two basis functions $\rightsquigarrow$ sum of entries

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Discretization of “Uniqueness Constraints”

$$\fint_\Omega u\mathbb{1} = \frac{1}{\int_\Omega \mathbb{1}} \int_\Omega u = 0$$

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$$\rightsquigarrow \frac{1}{M\vec{1}\cdot\vec{1}} M\vec{1}\cdot U = 0$$

where

M: mass matrix

$\vec{1}$: all-1 vector

U: discretization by point evaluation

Discretization of "Uniqueness Constraints"

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where
M: mass matrix
$\vec{1}$: all-1 vector
U: discretization by point evaluation

Let

$$\gimel := \frac{1}{M\vec{1}\cdot\vec{1}} M\vec{1}$$

Resulting System (Scalar Case)

Solve

$$L^{\#}U^{\#} = B^{\#}$$
$$\text{subject to } \gimel^{\#} \cdot U^{\#} = 0$$

with appropriate periodic vectors/matrices (later).

Resulting System (Vector-Valued Case)

Solve

$$\begin{bmatrix} E^{\#}_{00} & E^{\#}_{01} & E^{\#}_{02} \\ E^{\#}_{10} & E^{\#}_{11} & E^{\#}_{12} \\ E^{\#}_{20} & E^{\#}_{21} & E^{\#}_{22} \end{bmatrix} \begin{bmatrix} U^{\#}_0 \\ U^{\#}_1 \\ U^{\#}_2 \end{bmatrix} = \begin{bmatrix} B^{\#}_0 \\ B^{\#}_1 \\ B^{\#}_2 \end{bmatrix}$$

$$\text{subject to} \quad \begin{bmatrix} \gimel^{\#} \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} U^{\#}_0 \\ U^{\#}_1 \\ U^{\#}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \gimel^{\#} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} U^{\#}_0 \\ U^{\#}_1 \\ U^{\#}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \gimel^{\#} \end{bmatrix} \cdot \begin{bmatrix} U^{\#}_0 \\ U^{\#}_1 \\ U^{\#}_2 \end{bmatrix} = 0$$

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Projection onto Subspace Satisfying Constraint

Continuous: $s := \{u | \fint_\Omega u = 0\}$

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Discrete: $S := \mathrm{span}(\gimel)^\perp = \left\{ U \,\middle|\, \frac{1}{M\vec{1}\cdot\vec{1}} M\vec{1} \cdot U = 0 \right\}$

Projection onto Subspace Satisfying Constraint

Continuous: $s := \{u | \fint_\Omega u = 0\}$

$$\Pi_s u = u - \left(\fint_\Omega u \right) \mathbb{1}$$

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$$\Pi_S U = U - (\gimel \cdot U)\, \vec{1}$$

Projecting CG/PCG

- ensure that RHS satisfies constraint
- after each update, project destination vector (if sufficiently far off)

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Scalar Cell Problem: Poisson with $\fint_\Omega f = 0$

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1. set up vectors $\bar{U}, \tilde{F}, \vec{1}$ and matrices M, L
2. compute RHS $\quad B^\# := \natural(M\tilde{F}) - \natural(L\bar{U})$
3. periodize $\quad M^\# := \natural(M) \quad L^\# := \natural(L) \quad \vec{1}^\# := \sharp(\vec{1})$

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$$\gimel^\# = \frac{M^\#\vec{1}^\#}{M^\#\vec{1}^\# \cdot \vec{1}^\#}$$

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4. compute constraint vector

$$\gimel^\# = \frac{M^\#\vec{1}^\#}{M^\#\vec{1}^\# \cdot \vec{1}^\#}$$

5. solve the system

$$L^\#\tilde{U}^\# = B^\# = (M\tilde{F})^\# - (L\bar{U})^\#$$
$$\text{subject to } \gimel^\# \cdot \tilde{U}^\# = 0$$

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1. set up vectors $\bar{U}, \tilde{F}, \vec{1}$ and matrices M, L
2. compute RHS $\quad B^\# := \natural(M\tilde{F}) - \natural(L\bar{U})$
3. periodize $\quad M^\# := \natural(M) \quad L^\# := \natural(L) \quad \vec{1}^\# := \sharp(\vec{1})$
4. compute constraint vector

$$\gimel^\# = \frac{M^\#\vec{1}^\#}{M^\#\vec{1}^\# \cdot \vec{1}^\#}$$

5. solve the system

$$L^\#\tilde{U}^\# = B^\# = (M\tilde{F})^\# - (L\bar{U})^\#$$
$$\text{subject to } \gimel^\# \cdot \tilde{U}^\# = 0$$

6. periodically extend and add macroscopic part

$$U = \sharp^{-1}(\tilde{U}^\#) + \bar{U}$$

Vector-Valued Test Problem: Elasticity Without Source Term

1. set up $\bar{\mathbf{U}}, \vec{1}$, M, and elasticity block matrix $\mathbf{E}$

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Vector-Valued Test Problem: Elasticity Without Source Term

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4. compute constraint vectors

$$\gimel^{\#}_{\{0,1,2\}} = \frac{M^{\#}\vec{1}^{\#}}{M^{\#}\vec{1}^{\#} \cdot \vec{1}^{\#}} \left\{ \begin{bmatrix} \vec{1}^{\#} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \vec{1}^{\#} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \vec{1}^{\#} \end{bmatrix} \right\}$$

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$$\mathbf{E}^{\#}\tilde{\mathbf{U}}^{\#} = \mathbf{B}^{\#} = -(\mathbf{E}\bar{\mathbf{U}})^{\#}$$

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Vector-Valued Test Problem: Elasticity Without Source Term

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Contents

Basic Two-Grid Scheme

[Bakhalov; Fedorenko; Brandt 1977]

1. **Presmooth**: Perform ν_1 Gauß-Seidel iterations: $x^{(1)}$
2. Compute residual: $r = b - Ax^{(1)}$.
3. **Restrict** the residual: compute coarse version $\hat{r}$
4. **Solve** $\hat{A}\hat{e} = \hat{r}$
5. **Prolongate** the result: compute fine e
6. Add the coarse grid correction: $x^{(2)} = x^{(1)} + e$
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where smoothing reduces “high frequency components” and coarse grid correction reduces “low frequency components” of the error.

Multigrid for Cell Problems

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Modification of Prolongation

prolongation must respect new neighborhood relations

- loop over all nodes
- prolongate from facing present coarse node to proper fine neighbors

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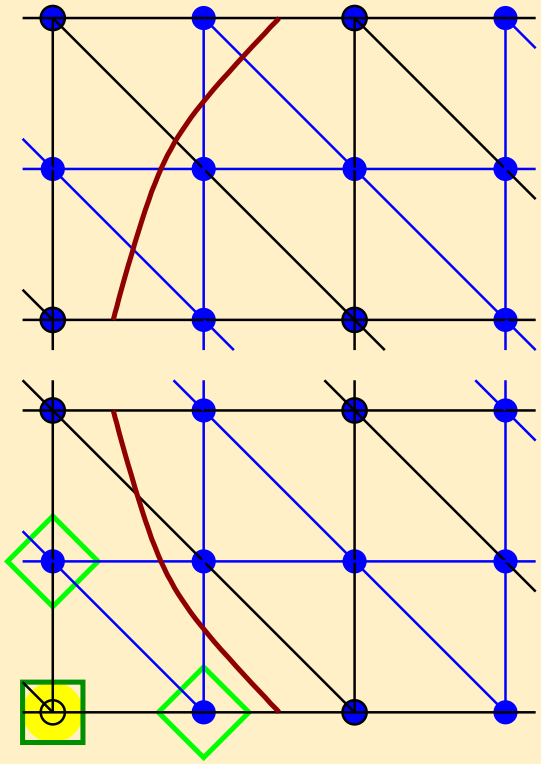

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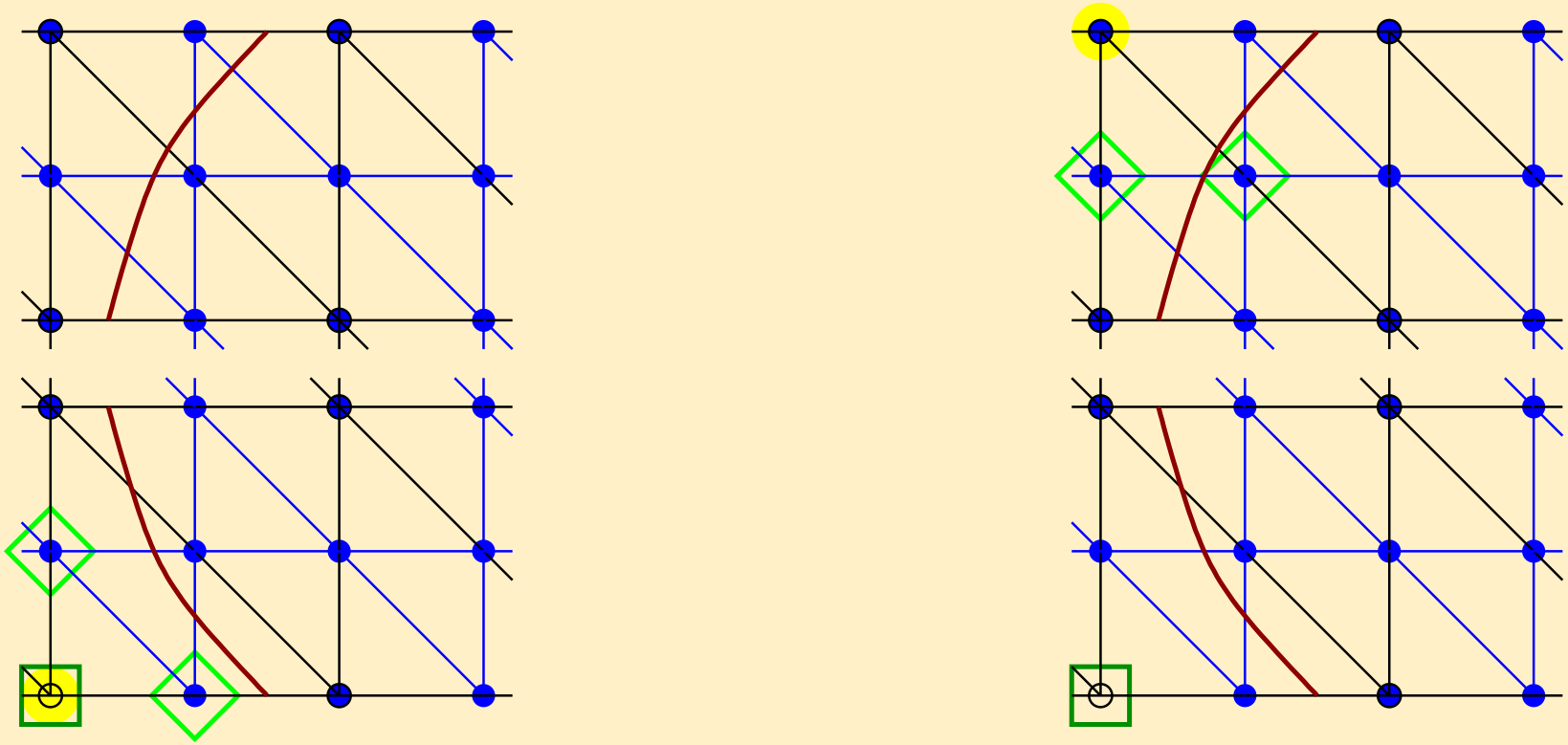

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- prolongate from facing present coarse node to proper fine neighbors

Restriction and Operator Coarsening

- Restriction $\mathcal{R} = \mathcal{P}^T$ is adjoint (transpose) of prolongation
- Coarse $\hat{A} = \mathcal{R}A\mathcal{P}$ as matrix product
- could compute $\hat{A}$ on the fly

Smoothing

Smoothing:

- only on presentDOFnodes
- pass bitMask $\rightsquigarrow$ more general selective smoother

Projection to “Uniqueness Constraints”

Smoothing and coarse-grid correction violate $\beth$ constraints.

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Projection to “Uniqueness Constraints”

Smoothing and coarse-grid correction violate $\daleth$ constraints.

- ✘ subspace projection after each iteration (on the finest grid only)
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- ✔ finest level only,
  - after presmoothing
  - after coarse-grid correction
  - after postsmoothing

Contents

Voigt's Notation

[Voigt]

Due to symmetry, replace

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- $3 \times 3 \times 3 \times 3$ tensor by 6×6 matrix

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$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix} = \begin{bmatrix} \star & \star & \star & & & \\ \star & \star & \star & & & \\ \star & \star & \star & & & \\ & & & \star & & \\ & & & & \star & \\ & & & & & \star \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{xy} \\ \epsilon_{xz} \\ \epsilon_{yz} \end{bmatrix}$$

$\star$ present if Poisson's ratio $\nu \neq 0$ ("bulging")

Resolution 33^3:

$$\begin{bmatrix} 1.482\times10^{+02} & 7.298\times10^{+01} & 7.298\times10^{+01} & 6.369\times10^{-20} & -2.123\times10^{-19} & 6.267\times10^{-29} \\ 7.298\times10^{+01} & 1.482\times10^{+02} & 7.298\times10^{+01} & -1.677\times10^{-18} & 1.111\times10^{-27} & 2.399\times10^{-18} \\ 7.298\times10^{+01} & 7.298\times10^{+01} & 1.482\times10^{+02} & 5.161\times10^{-32} & 1.599\times10^{-28} & 2.169\times10^{-27} \\ 3.256\times10^{-18} & 6.610\times10^{-18} & 3.256\times10^{-18} & 3.759\times10^{+01} & 3.764\times10^{-27} & -8.704\times10^{-19} \\ -2.649\times10^{-29} & 3.452\times10^{-29} & 1.959\times10^{-28} & -4.084\times10^{-29} & 3.759\times10^{+01} & -9.553\times10^{-20} \\ -2.843\times10^{-18} & -5.773\times10^{-18} & -2.843\times10^{-18} & -1.274\times10^{-18} & -2.301\times10^{-28} & 3.759\times10^{+01} \end{bmatrix}$$

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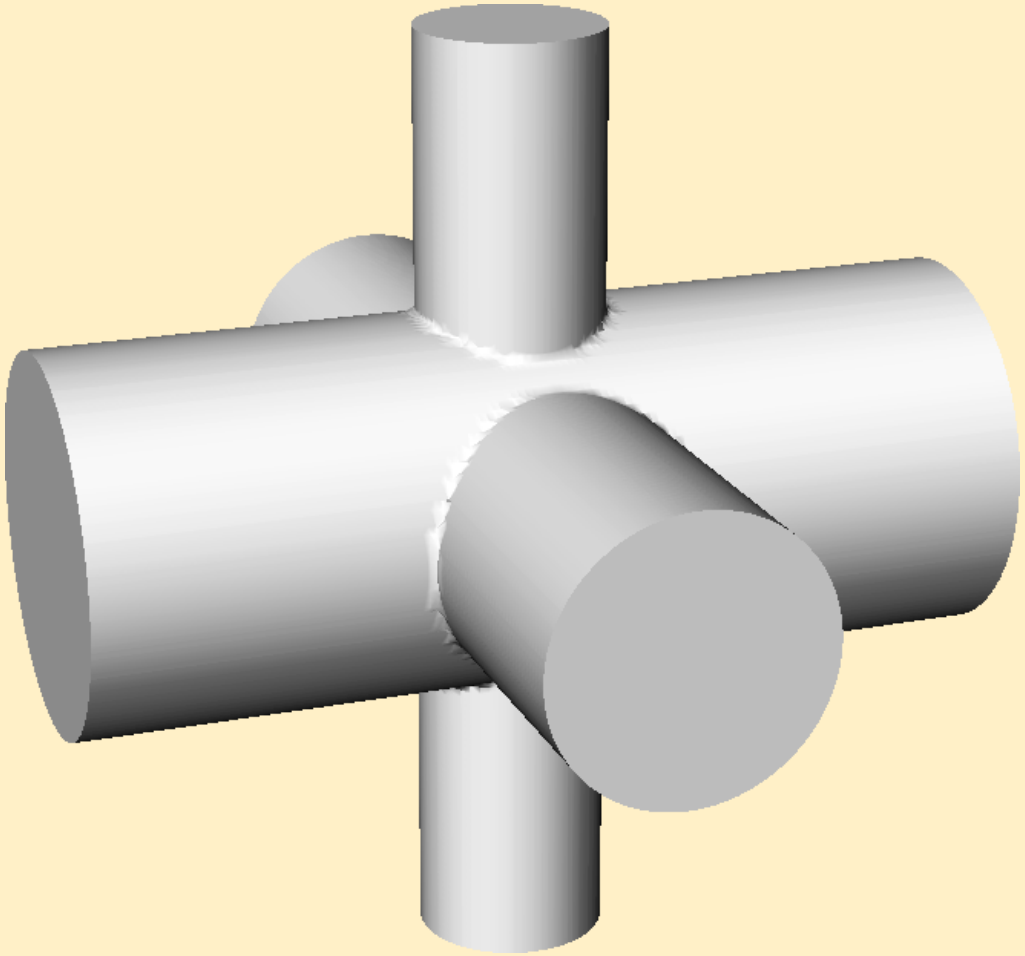
Resolution 65^3 / 129^3:

$$\begin{bmatrix} 1.482\times10^{+02} & 7.298\times10^{+01} & 7.298\times10^{+01} & 2.654\times10^{-21} & -1.858\times10^{-20} & 6.011\times10^{-29} \\ 7.298\times10^{+01} & 1.482\times10^{+02} & 7.298\times10^{+01} & 1.300\times10^{-19} & 6.838\times10^{-27} & 4.803\times10^{-19} \\ 7.298\times10^{+01} & 7.298\times10^{+01} & 1.482\times10^{+02} & 2.713\times10^{-30} & -1.507\times10^{-27} & 2.483\times10^{-27} \\ -2.060\times10^{-19} & -4.183\times10^{-19} & -2.060\times10^{-19} & 3.759\times10^{+01} & 1.847\times10^{-26} & 4.511\times10^{-20} \\ -9.038\times10^{-28} & 1.705\times10^{-27} & 2.888\times10^{-27} & -6.299\times10^{-28} & 3.759\times10^{+01} & -2.654\times10^{-21} \\ -1.597\times10^{-19} & -2.075\times10^{-19} & -2.764\times10^{-19} & 2.545\times10^{-30} & -3.052\times10^{-19} & 3.759\times10^{+01} \end{bmatrix}$$

$$\begin{bmatrix} 1.482\times10^{+02} & 7.298\times10^{+01} & 7.298\times10^{+01} & 6.634\times10^{-22} & -3.317\times10^{-21} & 8.553\times10^{-26} \\ 7.298\times10^{+01} & 1.482\times10^{+02} & 7.298\times10^{+01} & -1.231\times10^{-19} & 2.772\times10^{-26} & 7.695\times10^{-20} \\ 7.298\times10^{+01} & 7.298\times10^{+01} & 1.482\times10^{+02} & -1.031\times10^{-29} & -3.166\times10^{-26} & 3.514\times10^{-26} \\ 3.799\times10^{-20} & 7.713\times10^{-20} & 3.799\times10^{-20} & 3.759\times10^{+01} & 1.033\times10^{-25} & 1.493\times10^{-20} \\ -1.363\times10^{-26} & -5.848\times10^{-27} & -1.705\times10^{-26} & 8.934\times10^{-28} & 3.759\times10^{+01} & 4.644\times10^{-21} \\ 8.789\times10^{-20} & 1.728\times10^{-19} & 9.353\times10^{-20} & 3.914\times10^{-20} & 1.078\times10^{-20} & 3.759\times10^{+01} \end{bmatrix}$$

$1 \times 1 \times 1$ grid (0.4, 0.3, 0.2)



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Resolution 33^3:

$$\begin{bmatrix} 1.280 \times 10^{+01} & 1.191 \times 10^{+00} & 5.881 \times 10^{-01} & 1.429 \times 10^{-03} & -4.527 \times 10^{-03} & -2.823 \times 10^{-03} \\ 1.203 \times 10^{+00} & 7.617 \times 10^{+00} & 3.064 \times 10^{-01} & -5.823 \times 10^{-03} & 3.690 \times 10^{-03} & -3.170 \times 10^{-03} \\ 5.877 \times 10^{-01} & 3.137 \times 10^{-01} & 3.964 \times 10^{+00} & 1.563 \times 10^{-03} & -1.666 \times 10^{-03} & -5.369 \times 10^{-03} \\ 1.200 \times 10^{-03} & -2.507 \times 10^{-03} & 5.538 \times 10^{-03} & 7.038 \times 10^{-01} & 3.277 \times 10^{-03} & -2.520 \times 10^{-03} \\ -4.967 \times 10^{-03} & 3.036 \times 10^{-03} & -1.412 \times 10^{-03} & 2.718 \times 10^{-03} & 2.691 \times 10^{-01} & -7.311 \times 10^{-03} \\ -2.770 \times 10^{-03} & -6.013 \times 10^{-03} & -6.026 \times 10^{-03} & -2.042 \times 10^{-03} & -7.838 \times 10^{-03} & 1.768 \times 10^{-01} \end{bmatrix}$$

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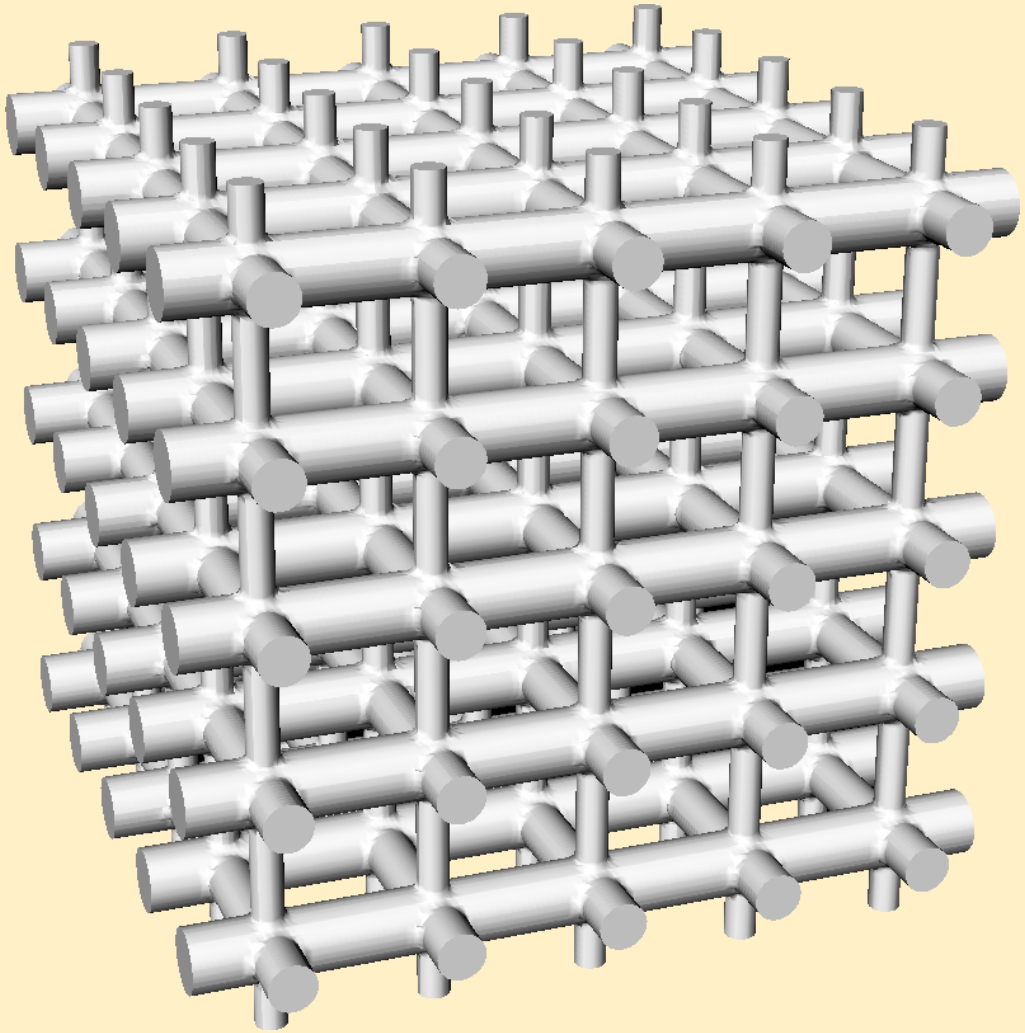
Resolution 129^3:

$$\begin{bmatrix} 1.324\times10^{+01} & 1.221\times10^{+00} & 5.717\times10^{-01} & -4.006\times10^{-04} & -6.436\times10^{-04} & -3.714\times10^{-04} \\ 1.226\times10^{+00} & 8.143\times10^{+00} & 3.551\times10^{-01} & -1.470\times10^{-03} & 3.603\times10^{-04} & -6.363\times10^{-04} \\ 5.751\times10^{-01} & 3.553\times10^{-01} & 3.922\times10^{+00} & 2.991\times10^{-04} & -3.021\times10^{-04} & -9.727\times10^{-04} \\ -3.994\times10^{-04} & -1.272\times10^{-03} & 5.489\times10^{-04} & 6.709\times10^{-01} & 3.461\times10^{-04} & -2.217\times10^{-04} \\ -6.440\times10^{-04} & 3.850\times10^{-04} & -2.499\times10^{-04} & 3.404\times10^{-04} & 2.222\times10^{-01} & -8.796\times10^{-04} \\ -3.419\times10^{-04} & -6.335\times10^{-04} & -1.036\times10^{-03} & -1.937\times10^{-04} & -8.180\times10^{-04} & 1.733\times10^{-01} \end{bmatrix}$$

$5 \times 5 \times 5$ / $10 \times 10 \times 10$ grid (0.4, 0.3, 0.2)



$5\times5\times5$ / $10\times10\times10$ grid (0.4, 0.3, 0.2)

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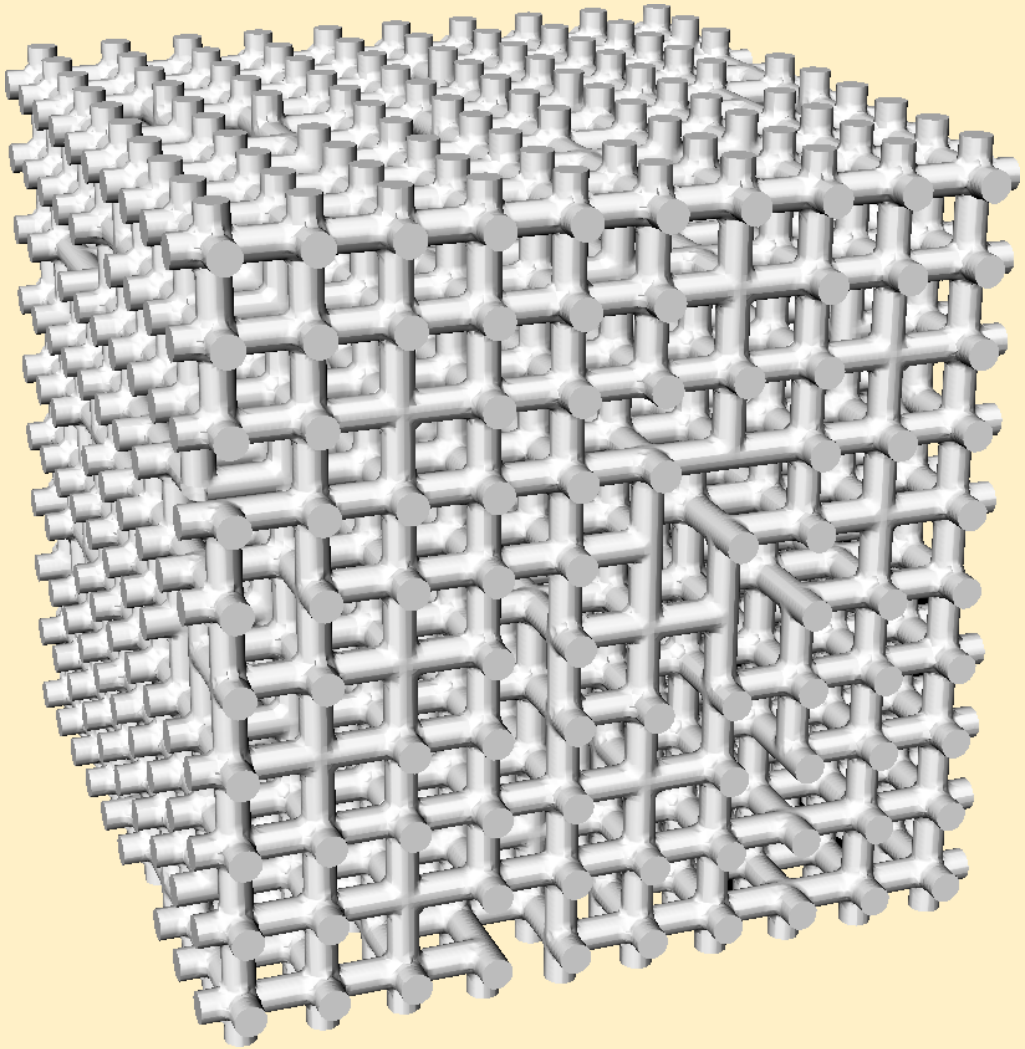
$5 \times 5 \times 5$, resolution 129^3:

$$\begin{bmatrix} 1.298\times10^{+01} & 1.226\times10^{+00} & 5.555\times10^{-01} & 1.848\times10^{-03} & -7.020\times10^{-03} & -3.232\times10^{-03} \\ 1.245\times10^{+00} & 7.952\times10^{+00} & 3.141\times10^{-01} & -6.921\times10^{-03} & 4.621\times10^{-03} & -9.158\times10^{-03} \\ 5.726\times10^{-01} & 3.204\times10^{-01} & 3.762\times10^{+00} & 2.922\times10^{-03} & -2.946\times10^{-03} & -8.226\times10^{-03} \\ 2.476\times10^{-03} & -6.149\times10^{-03} & 5.753\times10^{-03} & 7.424\times10^{-01} & 4.037\times10^{-03} & -4.090\times10^{-03} \\ -6.115\times10^{-03} & 4.485\times10^{-03} & -2.672\times10^{-03} & 4.355\times10^{-03} & 2.667\times10^{-01} & -1.105\times10^{-02} \\ -5.246\times10^{-03} & -9.233\times10^{-03} & -1.069\times10^{-02} & -3.422\times10^{-03} & -1.103\times10^{-02} & 1.933\times10^{-01} \end{bmatrix}$$

$10 \times 10 \times 10$, resolution 129^3:

$$\begin{bmatrix} 1.268\times10^{+01} & 1.270\times10^{+00} & 6.273\times10^{-01} & 3.492\times10^{-03} & -1.221\times10^{-02} & -1.716\times10^{-02} \\ 1.252\times10^{+00} & 8.001\times10^{+00} & 3.548\times10^{-01} & -1.348\times10^{-02} & 1.419\times10^{-02} & -4.911\times10^{-02} \\ 5.742\times10^{-01} & 3.369\times10^{-01} & 4.006\times10^{+00} & 6.653\times10^{-03} & -2.613\times10^{-03} & -3.951\times10^{-02} \\ 1.053\times10^{-03} & -1.199\times10^{-02} & 1.106\times10^{-02} & 8.864\times10^{-01} & 5.905\times10^{-03} & -9.328\times10^{-03} \\ -7.262\times10^{-03} & 1.464\times10^{-02} & -8.707\times10^{-04} & 4.689\times10^{-03} & 3.875\times10^{-01} & -3.902\times10^{-02} \\ -2.635\times10^{-02} & -4.355\times10^{-02} & -4.455\times10^{-02} & -8.633\times10^{-03} & -3.529\times10^{-02} & 2.838\times10^{-01} \end{bmatrix}$$

$10 \times 10 \times 10$ grid (0.4, 0.4, 0.4), 8 % removed



$10 \times 10 \times 10$ grid (0.4, 0.4, 0.4), 8 % removed

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Samples 0 . . . 2:

$$\begin{bmatrix} 9.970 \times 10^{+00} & 1.124 \times 10^{+00} & 1.225 \times 10^{+00} & 1.410 \times 10^{-03} & -1.842 \times 10^{-02} & -1.597 \times 10^{-02} \\ 1.108 \times 10^{+00} & 1.013 \times 10^{+01} & 1.227 \times 10^{+00} & -7.125 \times 10^{-04} & 1.875 \times 10^{-02} & 2.546 \times 10^{-02} \\ 1.188 \times 10^{+00} & 1.206 \times 10^{+00} & 1.092 \times 10^{+01} & 1.835 \times 10^{-02} & -8.332 \times 10^{-03} & 3.833 \times 10^{-02} \\ 3.230 \times 10^{-03} & 1.892 \times 10^{-03} & 1.784 \times 10^{-02} & 1.153 \times 10^{+00} & 2.450 \times 10^{-02} & -1.276 \times 10^{-02} \\ -2.114 \times 10^{-03} & 3.682 \times 10^{-02} & 2.198 \times 10^{-03} & 2.192 \times 10^{-02} & 1.280 \times 10^{+00} & -4.114 \times 10^{-02} \\ 2.205 \times 10^{-03} & 4.481 \times 10^{-02} & 4.214 \times 10^{-02} & -8.554 \times 10^{-03} & -3.413 \times 10^{-02} & 1.272 \times 10^{+00} \end{bmatrix}$$

$$\begin{bmatrix} 1.105 \times 10^{+01} & 1.301 \times 10^{+00} & 1.283 \times 10^{+00} & -1.696 \times 10^{-02} & 7.812 \times 10^{-03} & -2.865 \times 10^{-02} \\ 1.290 \times 10^{+00} & 1.135 \times 10^{+01} & 1.335 \times 10^{+00} & -1.228 \times 10^{-02} & 1.909 \times 10^{-02} & -2.182 \times 10^{-02} \\ 1.268 \times 10^{+00} & 1.333 \times 10^{+00} & 1.130 \times 10^{+01} & 1.784 \times 10^{-02} & -3.443 \times 10^{-03} & -1.283 \times 10^{-02} \\ -1.896 \times 10^{-02} & -1.364 \times 10^{-02} & 1.665 \times 10^{-02} & 1.364 \times 10^{+00} & 1.959 \times 10^{-02} & -1.586 \times 10^{-02} \\ 8.637 \times 10^{-03} & 1.973 \times 10^{-02} & 7.709 \times 10^{-04} & 1.940 \times 10^{-02} & 1.405 \times 10^{+00} & -3.193 \times 10^{-02} \\ -2.926 \times 10^{-02} & -1.969 \times 10^{-02} & -1.555 \times 10^{-02} & -1.358 \times 10^{-02} & -2.818 \times 10^{-02} & 1.402 \times 10^{+00} \end{bmatrix}$$

$$\begin{bmatrix} 1.113 \times 10^{+01} & 1.329 \times 10^{+00} & 1.346 \times 10^{+00} & -1.677 \times 10^{-02} & -5.960 \times 10^{-03} & -2.795 \times 10^{-02} \\ 1.323 \times 10^{+00} & 1.139 \times 10^{+01} & 1.344 \times 10^{+00} & -1.222 \times 10^{-02} & 2.264 \times 10^{-02} & -8.942 \times 10^{-03} \\ 1.337 \times 10^{+00} & 1.342 \times 10^{+00} & 1.150 \times 10^{+01} & 2.080 \times 10^{-02} & 2.389 \times 10^{-03} & 8.162 \times 10^{-03} \\ -1.828 \times 10^{-02} & -1.703 \times 10^{-02} & 1.464 \times 10^{-02} & 1.373 \times 10^{+00} & 2.141 \times 10^{-02} & -1.187 \times 10^{-02} \\ -3.293 \times 10^{-03} & 2.832 \times 10^{-02} & 1.059 \times 10^{-02} & 2.131 \times 10^{-02} & 1.415 \times 10^{+00} & -3.143 \times 10^{-02} \\ -2.917 \times 10^{-02} & -6.804 \times 10^{-03} & 5.230 \times 10^{-03} & -1.006 \times 10^{-02} & -2.723 \times 10^{-02} & 1.404 \times 10^{+00} \end{bmatrix}$$

$10 \times 10 \times 10$ grid (0.4, 0.4, 0.4), 8 % removed

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Samples 3 . . . 5:

$$\begin{bmatrix} 1.109 \times 10^{+01} & 1.318 \times 10^{+00} & 1.311 \times 10^{+00} & -3.386 \times 10^{-02} & 5.309 \times 10^{-03} & -2.370 \times 10^{-02} \\ 1.304 \times 10^{+00} & 1.148 \times 10^{+01} & 1.333 \times 10^{+00} & -2.277 \times 10^{-02} & 1.908 \times 10^{-02} & 1.787 \times 10^{-02} \\ 1.301 \times 10^{+00} & 1.336 \times 10^{+00} & 1.136 \times 10^{+01} & 1.695 \times 10^{-02} & 1.306 \times 10^{-02} & 1.445 \times 10^{-02} \\ -3.665 \times 10^{-02} & -2.680 \times 10^{-02} & 1.218 \times 10^{-02} & 1.384 \times 10^{+00} & 2.059 \times 10^{-02} & -1.320 \times 10^{-02} \\ 9.049 \times 10^{-03} & 2.383 \times 10^{-02} & 2.021 \times 10^{-02} & 2.036 \times 10^{-02} & 1.411 \times 10^{+00} & -3.444 \times 10^{-02} \\ -2.325 \times 10^{-02} & 2.180 \times 10^{-02} & 1.237 \times 10^{-02} & -1.107 \times 10^{-02} & -3.041 \times 10^{-02} & 1.408 \times 10^{+00} \end{bmatrix}$$

$$\begin{bmatrix} 1.118 \times 10^{+01} & 1.298 \times 10^{+00} & 1.301 \times 10^{+00} & -7.678 \times 10^{-03} & -2.434 \times 10^{-02} & -2.085 \times 10^{-02} \\ 1.283 \times 10^{+00} & 1.137 \times 10^{+01} & 1.322 \times 10^{+00} & 6.421 \times 10^{-03} & 1.423 \times 10^{-02} & 1.636 \times 10^{-02} \\ 1.286 \times 10^{+00} & 1.326 \times 10^{+00} & 1.139 \times 10^{+01} & 1.954 \times 10^{-02} & -1.752 \times 10^{-02} & 3.517 \times 10^{-02} \\ -8.899 \times 10^{-03} & 3.187 \times 10^{-03} & 1.840 \times 10^{-02} & 1.395 \times 10^{+00} & 2.195 \times 10^{-02} & -1.922 \times 10^{-02} \\ -2.651 \times 10^{-02} & 1.027 \times 10^{-02} & -1.625 \times 10^{-02} & 2.286 \times 10^{-02} & 1.414 \times 10^{+00} & -3.168 \times 10^{-02} \\ -2.267 \times 10^{-02} & 1.944 \times 10^{-02} & 3.324 \times 10^{-02} & -1.766 \times 10^{-02} & -2.691 \times 10^{-02} & 1.424 \times 10^{+00} \end{bmatrix}$$

$$\begin{bmatrix} 1.109 \times 10^{+01} & 1.280 \times 10^{+00} & 1.362 \times 10^{+00} & -1.591 \times 10^{-02} & 8.770 \times 10^{-03} & -2.642 \times 10^{-02} \\ 1.272 \times 10^{+00} & 1.118 \times 10^{+01} & 1.320 \times 10^{+00} & -9.916 \times 10^{-03} & 1.902 \times 10^{-02} & -6.589 \times 10^{-03} \\ 1.348 \times 10^{+00} & 1.319 \times 10^{+00} & 1.152 \times 10^{+01} & 1.795 \times 10^{-02} & -5.610 \times 10^{-03} & 2.547 \times 10^{-05} \\ -1.786 \times 10^{-02} & -9.456 \times 10^{-03} & 1.824 \times 10^{-02} & 1.359 \times 10^{+00} & 2.009 \times 10^{-02} & -1.589 \times 10^{-02} \\ 9.584 \times 10^{-03} & 2.120 \times 10^{-02} & -7.434 \times 10^{-04} & 2.013 \times 10^{-02} & 1.417 \times 10^{+00} & -3.282 \times 10^{-02} \\ -2.636 \times 10^{-02} & -2.770 \times 10^{-03} & -1.615 \times 10^{-03} & -1.400 \times 10^{-02} & -2.912 \times 10^{-02} & 1.390 \times 10^{+00} \end{bmatrix}$$

Contents

Postprocessing of Tensors

- axis alignment?
- visualization

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- let multigrid coarsening “automagically” treat non-pointwise periodic BC??

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