Mulitgrid Methods

Achi Brandt:

Multi-Level Adaptive Solutions to Boundary Value Problems

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Lars O. Schwen

Outline

- Review: Computational Complexity
- The Idea of Multigrid
- Algorithm & Properties
- Grid Management
- The Author & Some History of MG

Review: Computational Complexity

$$\Delta u = u_{xx} \qquad\qquad = f_{\mathsf{rhs}}, m^1 \quad \text{grid.}$$

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$$\Delta u = u_{xx} + u_{yy} \qquad = f_{\mathrm{rhs}}, m^{1,2} \quad \text{grid.}$$

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- Conjugate gradient (to fixed tolerance: $O(m)$ iterations):

  $O(m^{2,3,4})$

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- want to solve on G^M

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- Solving the problem $\Leftrightarrow$ reducing the residuals V^k:

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- Function resolved on coarser grid will lose high frequencies (Nyquist)

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- Iterate until on G^0, use cheap non–sweeping method

Code: decay of high frequencies

An Example

Problem $LU = U_{xx} + cU_{yy} = F$ with "suitable BC".

- Standard second order discretization:

$$L^k U^k = \frac{U^k_{\alpha+1,\beta} - 2U^k_{\alpha\beta} + U^k_{\alpha-1,\beta}}{h_k^2} + c \cdot \frac{U^k_{\alpha,\beta+1} - 2U^k_{\alpha\beta} + U^k_{\alpha,\beta-1}}{h_k^2} = F^k_{\alpha,\beta} \tag{6}$$

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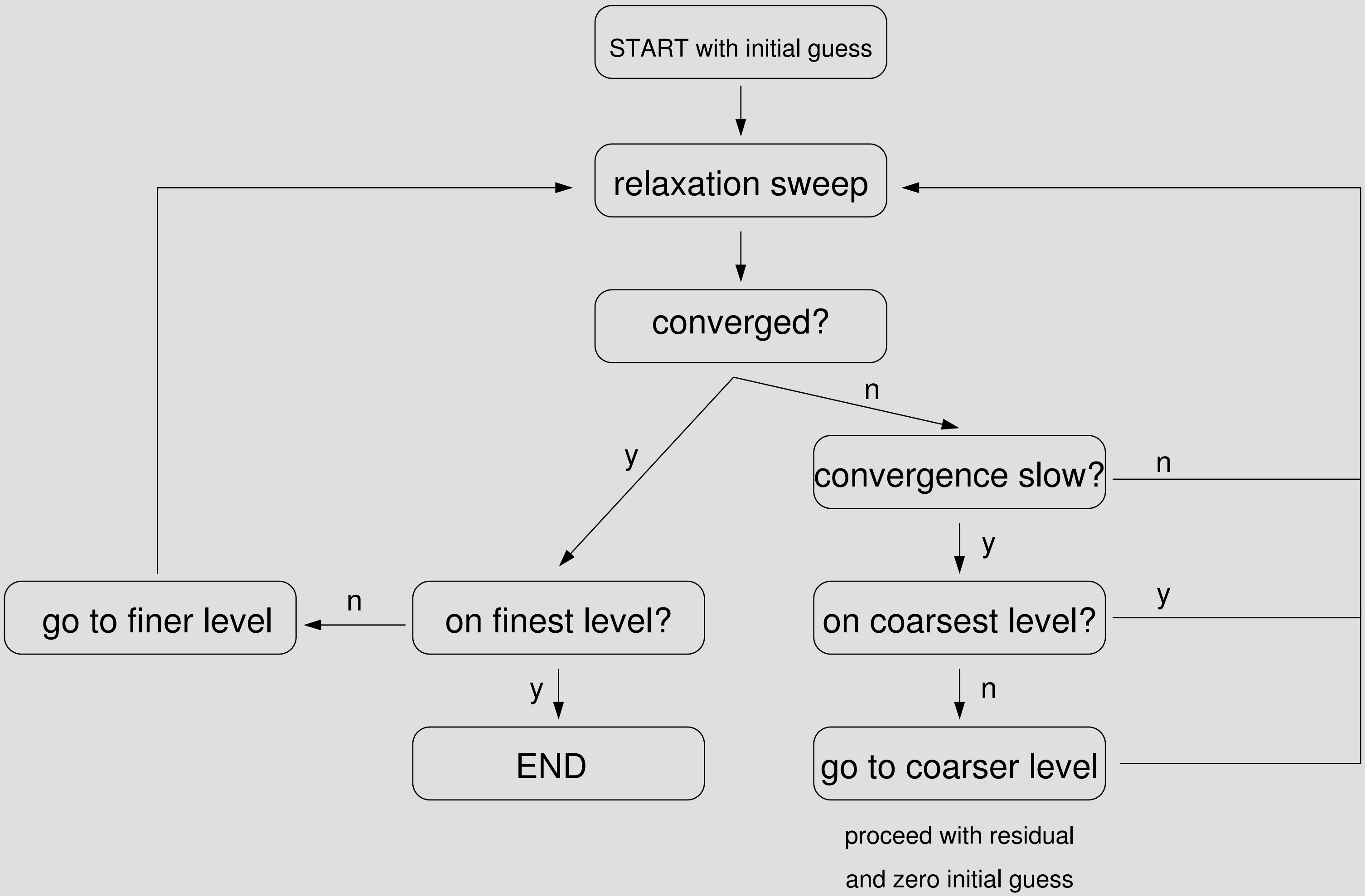
- Do not need to use relaxation parameter

$$\omega \neq 1 : u \leftarrow u + \omega(\bar{u} - u) \tag{14}$$

in general cannot do significantly better than $\omega = 1$.

C-Cycle


START with initial guess
relaxation sweep
converged?
n
y
convergence slow?
n
y
on coarsest level?
y
n
go to finer level
n
on finest level?
y
END
go to coarser level
proceed with residual
and zero initial guess

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- How to go to finer level (interpolation)?

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- Better: smooth residuals, e. g. high order interpolation of solution on G^{M-1}.
- Try to preserve smoothness in the beginning (later)

Convergence and Slow Convergence

Criterion when more improvement per work by continuing on finer grid, assuming that error components $|\theta| \approx \frac{\pi}{2}$ dominate:

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Criterion when convergence has slowed down for high frequencies:

$$\frac{\|\text{residual}\|}{\|\text{residual one step before}\|} \geq \eta = \frac{1+3\bar{\mu}}{4} \tag{16}$$

for an appropriate grid and I_k^{k-1}-weighting. $\bar{\mu}$ is the max. smoothing factor for frequencies for which coarse grid correction is not effective.

- If $\eta \not\equiv$ const on Ω, choose maximal η

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- E. g. Poisson + Gauß–Seidel:
  - ⋆ $\delta = 0.219, \eta = 0.625$
  - ⋆ for smaller η similarly good results
  - ⋆ for $\eta \leq 0.95$ maximal double work
  - ⋆ for $0.0001 \leq \delta \leq 0.7$ maximal double work

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- Later higher order is not more efficient (not significantly better, more computational work)
- FE: Structure determines how to interpolate/coarsen.

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- For difference equations with rapidly varying coefficients $\rho_\nu = 2^{-d-|\nu|_\infty}$ for $|\nu|_\infty \leq 1$.

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$$2n \cdot \left(1 + 2^{-1d} + 2^{-2d} + \dots\right) \quad \leq \quad 2^n \cdot \frac{2^d}{2^d - 1} \tag{18}$$

Code: C-Cycle

FAS

Full Approximation Storage mode of C-Cycle

- Above: store v^k designed to correct finer level u^{k+1}.
- FAS: store full current approximation $u^k = I_k^{k+1} u^{k+1} + v^k$.

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- Can also handle nonlinear problems
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- Get good estimate for truncation error, same approximation behavior as for solution

Nonuniform Grids

- Idea: adaptive grids, openended sequence $G^0, \ldots, G^m$, non-coextensive
  - ⋆ finer grids on increasingly smaller subdomains: adaption where needed
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- Also need to keep track of coordinates of grid points and adjacency

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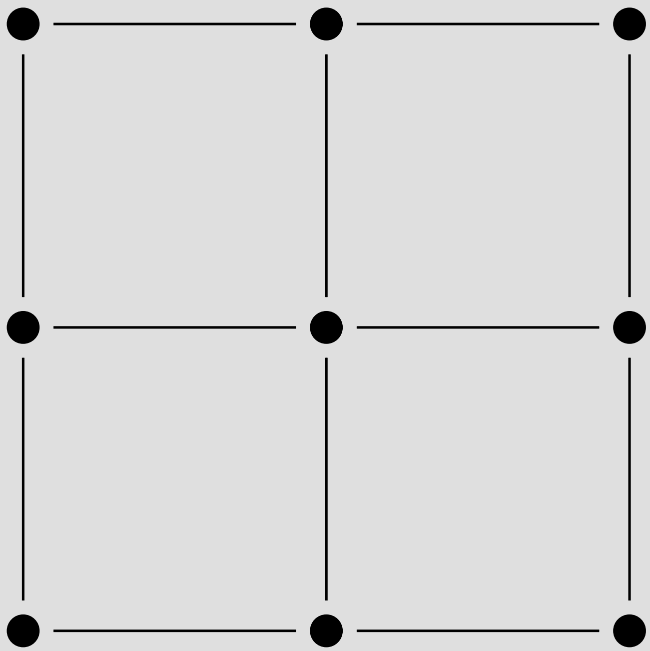

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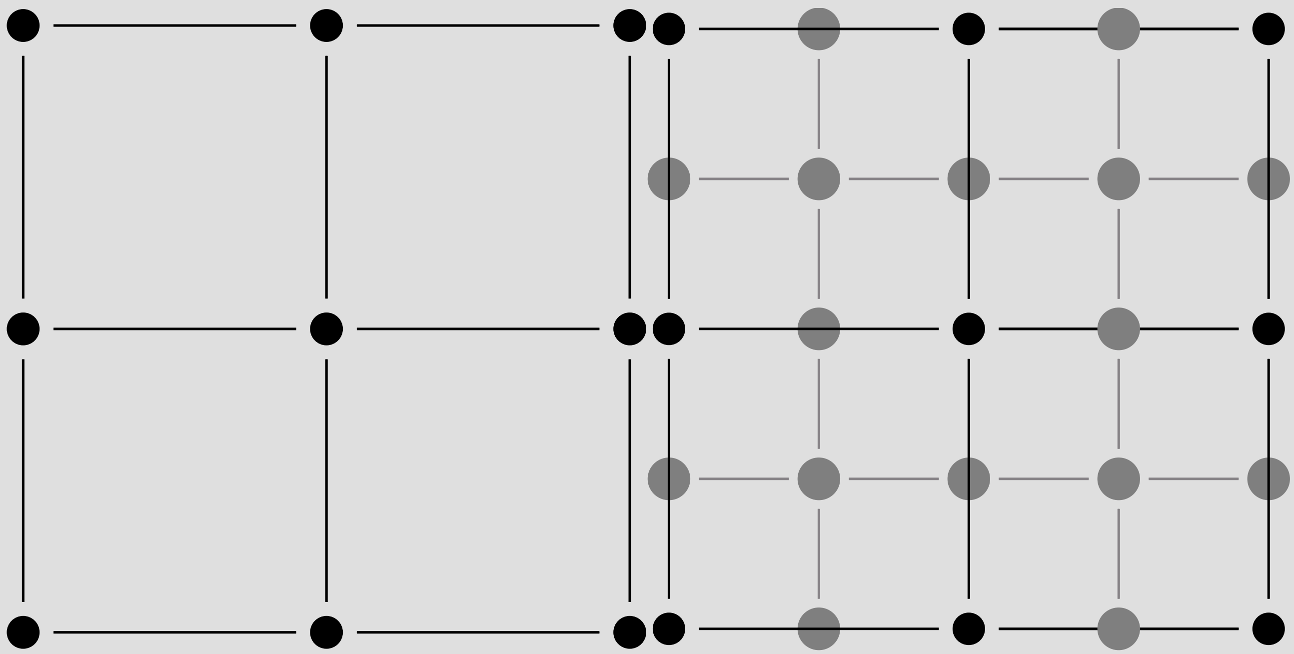

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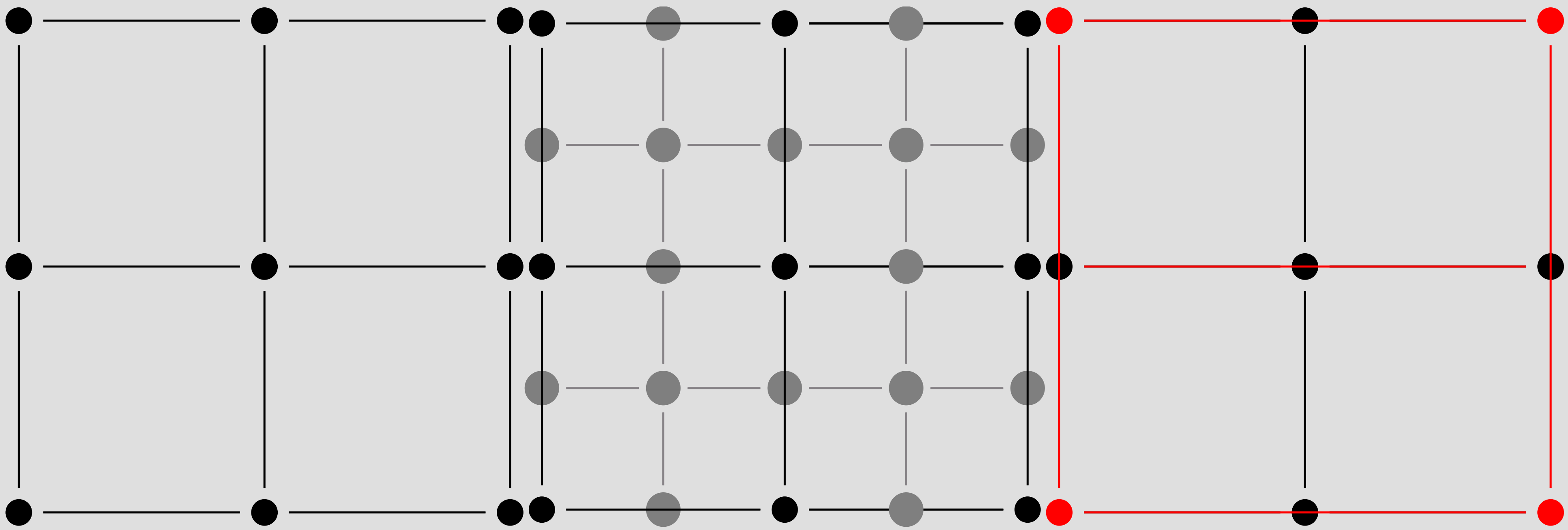
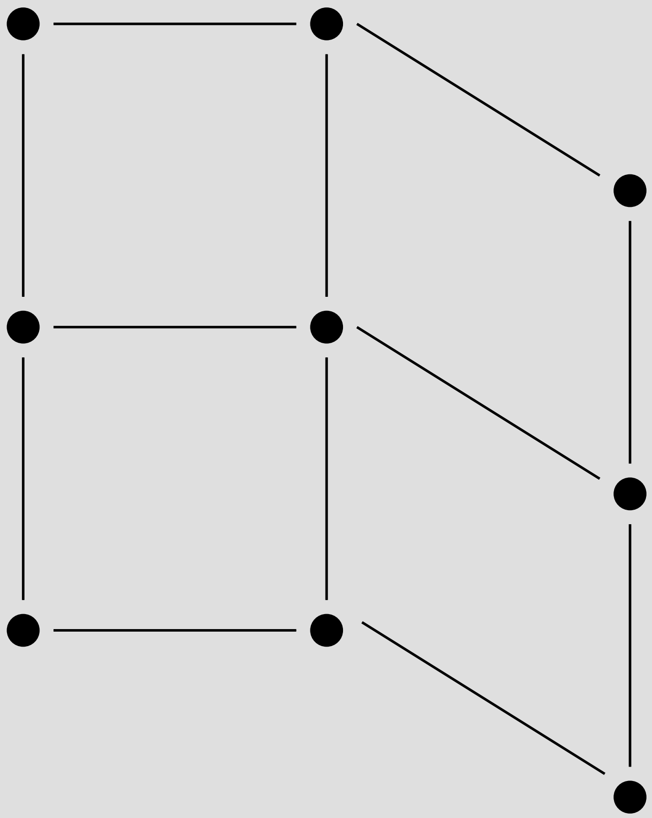


“Non-complicated” Grids



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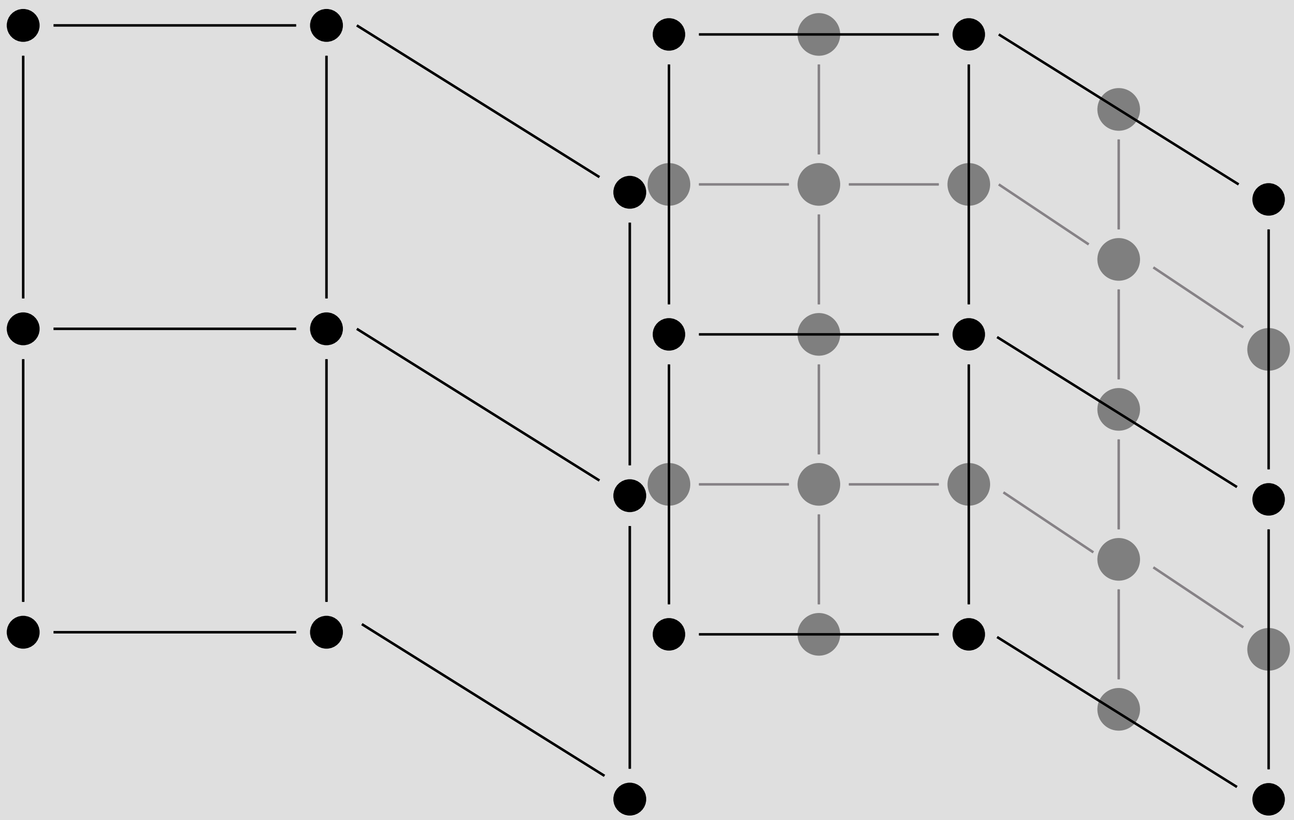

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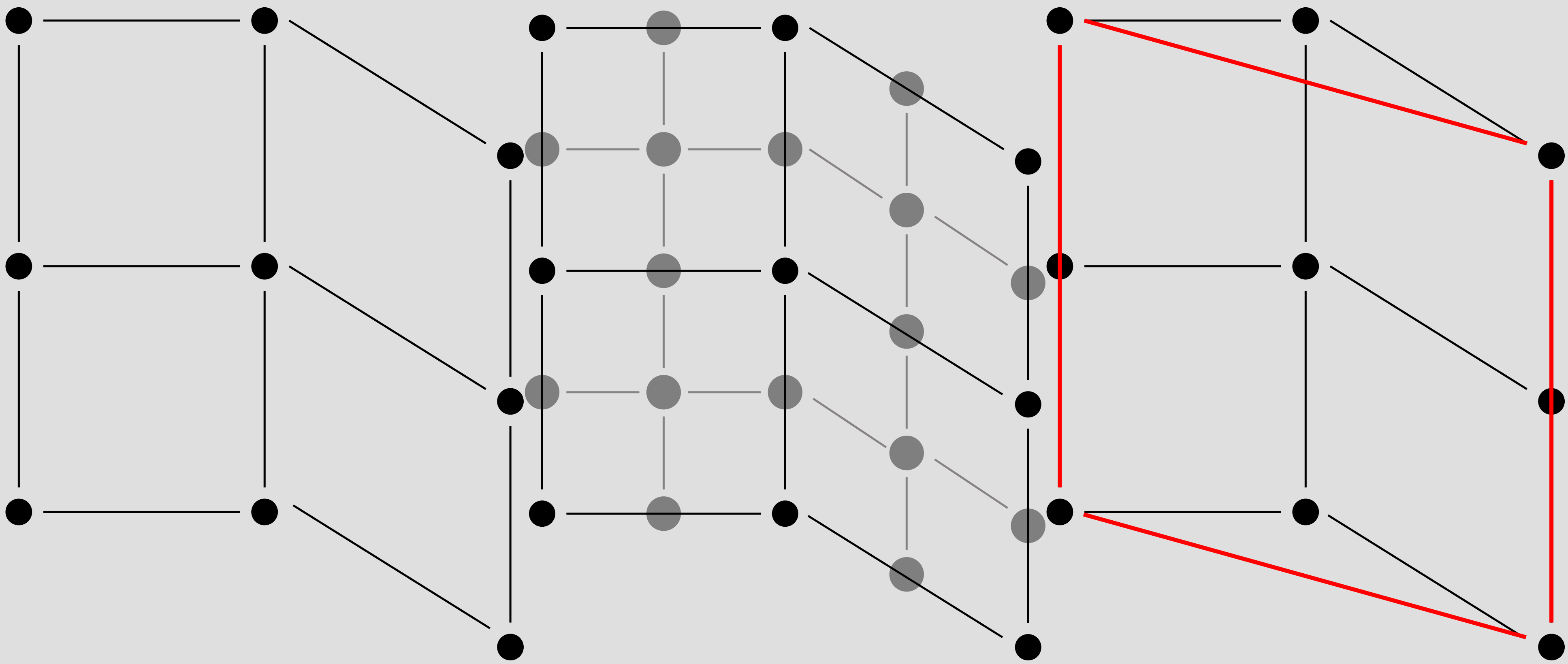
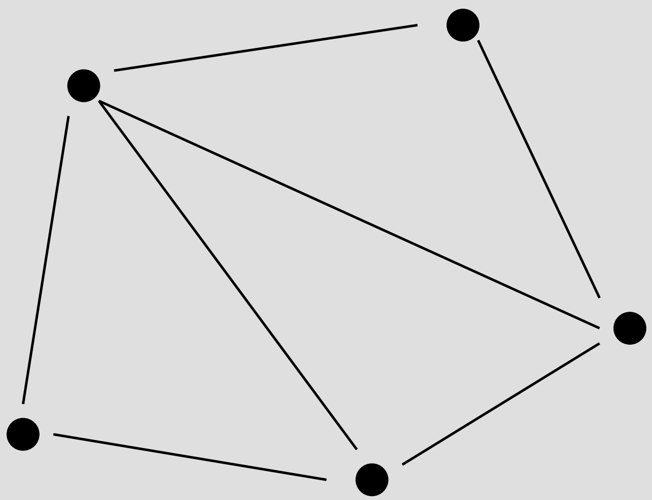


“Complicated” Grids



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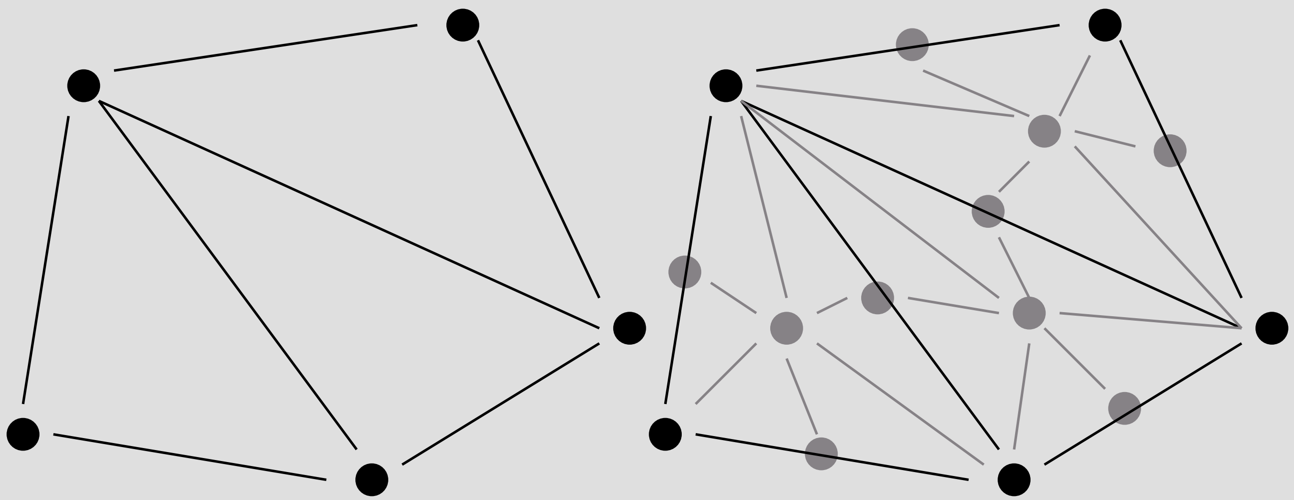

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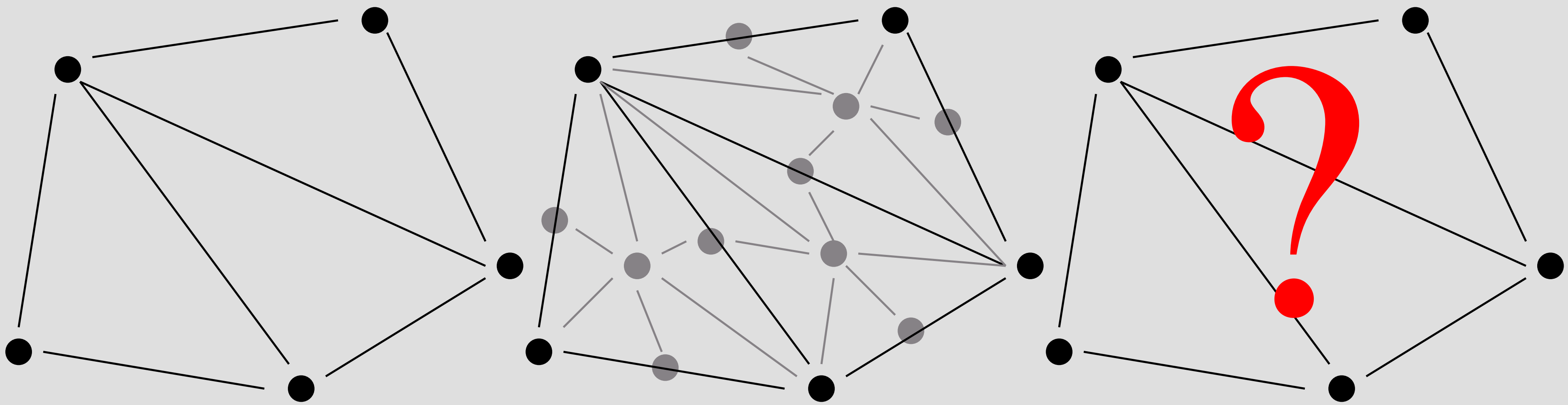

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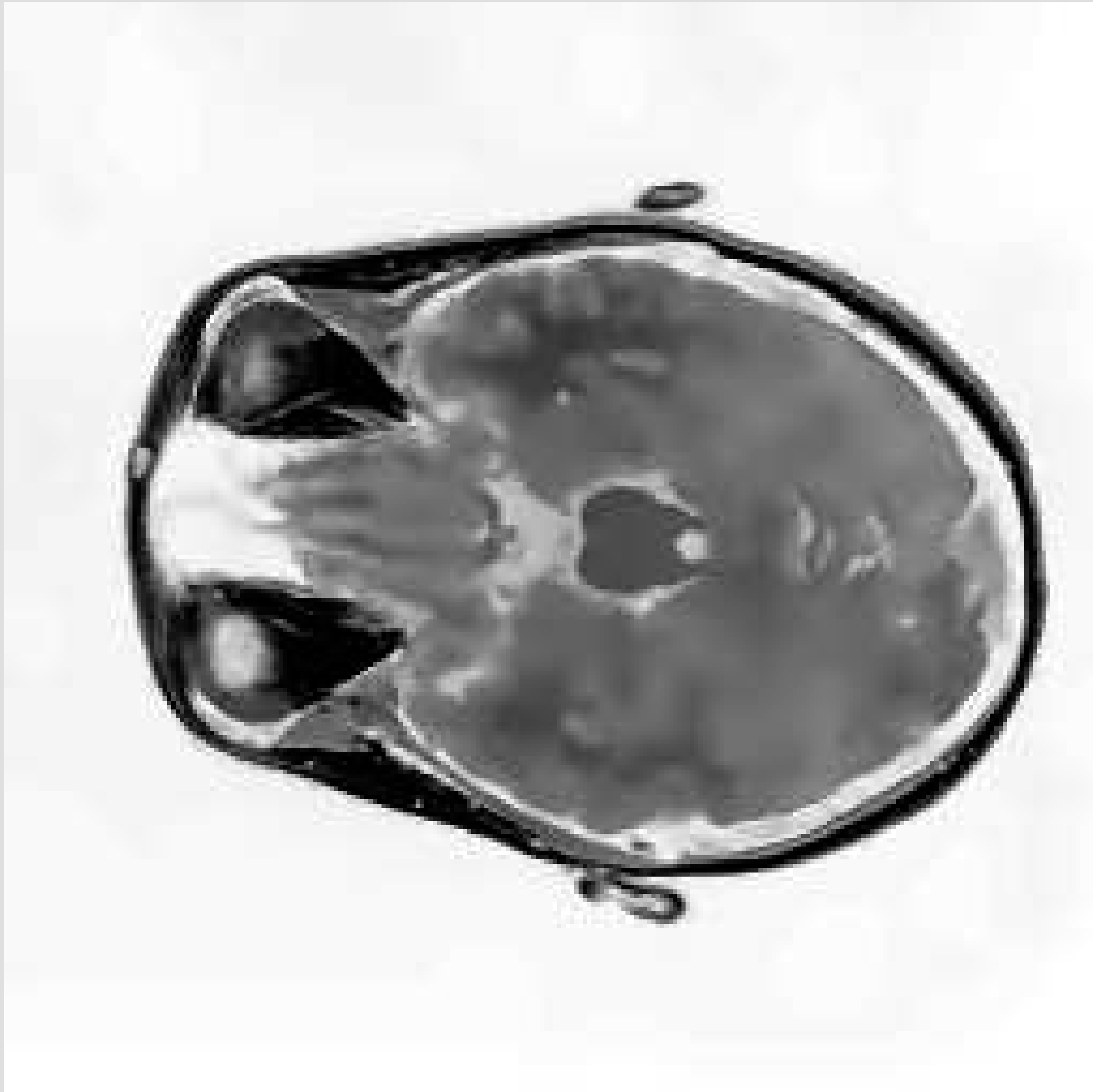
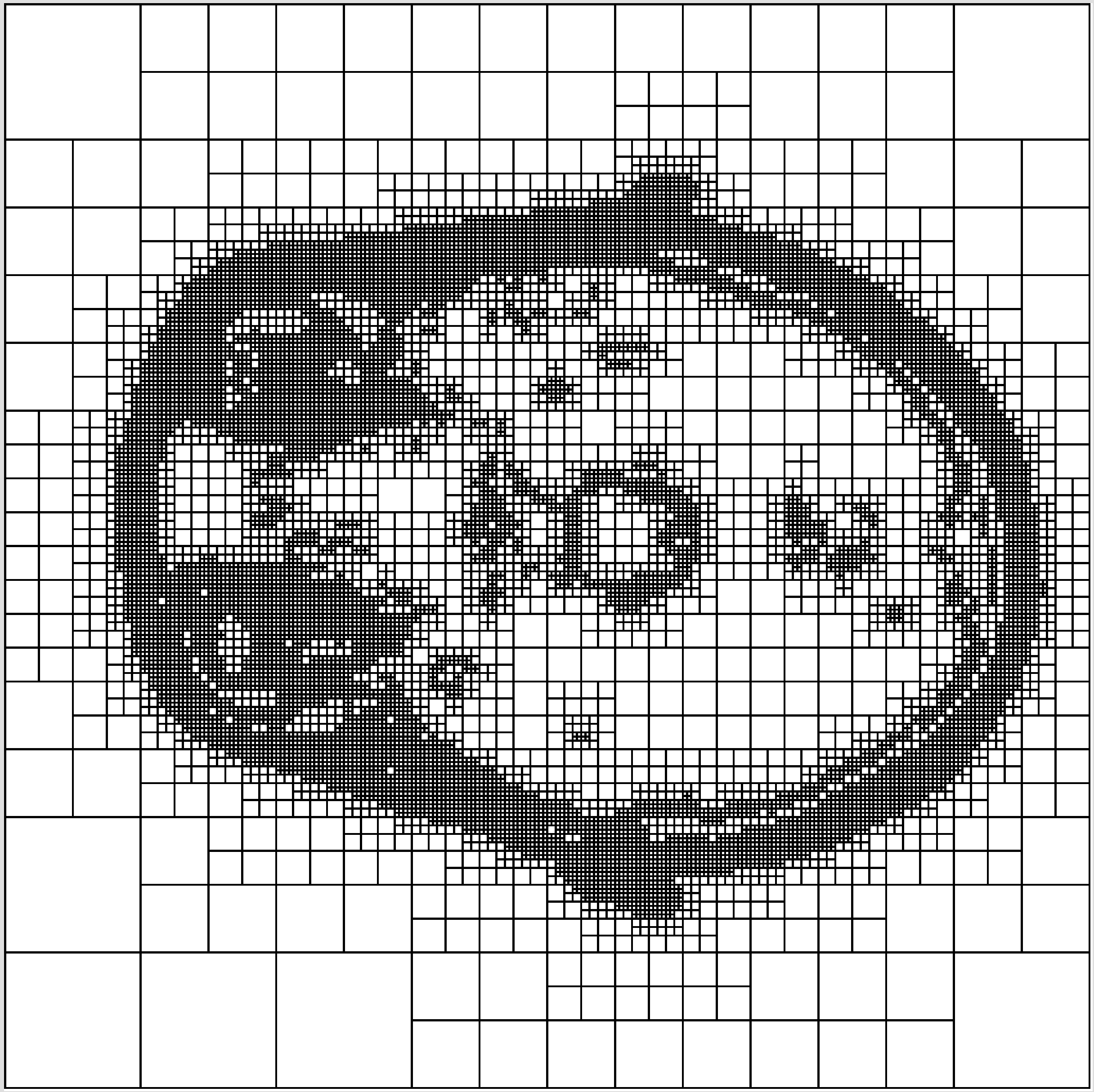
- Hierarchy of uniform rectangular grids
- Coarsest grid covers entire domain
- Finer grids introduced where necessary
  - ⋆ they will cover non-connected subdomains.
  - ⋆ need appropriate memory management for efficient computation
  - ⋆ good because calculations can be done independent of other subdomains (parallel computing)
  - ⋆ easy calculations on uniform grids

Example for adaptive grid refinement:





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Images from: T. Preußer, M. Rumpf: An Adaptive Finite Element Method for Large Scale Image Processing. Journal of Visual Comm. and Image Repres., 11, pp. 183-195, 2000.

The original data set does not contain jpeg artefacts, unlike the image in the pdf version of the article.

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Computational work

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Work to reduce error on finest grid to order of truncation error:

$$W_M \leq \frac{\left(1 + \hat{\rho}^d + \hat{\rho}^{2d} + \ldots\right) p \log \hat{\rho}}{\log \mathring{\mu}} \tag{19}$$

$$\leq \frac{p \log \hat{\rho}}{\left(1 - \hat{\rho}^d\right)^2 \log \bar{\mu}} \tag{20}$$

where $\bar{\mu} = \max\limits_{\hat{\rho} \leq |\theta| \leq \pi} \mu(\theta)$ maximal smoothing factor, $\mathring{\mu} = \bar{\mu}^{(1-\hat{\rho}^d)}$ multigrid convergence factor, p order of interpolation, d dimension $\hat{\rho}$ mesh size ratio.

Optimization of h, p

In general, can improve by reducing the mesh size h and by increasing the order of approximation p. View this as an optimization problem:

Minimize error (estimator) for a given computational work or vice versa, usually $-\log E \sim W$.

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$$E = \int_\Omega G(x)\tau(x)\,\mathrm{d}x \tag{21}$$

where

- $\tau(x) = |LU(x) - L^h U(x)|$ truncation error
- $G(x)$ appropriate weighting, e. g. $G(x) = \left[\mathrm{dist}(x, \partial\Omega)\right]^{m/2-l}$ for mth order elliptic problem, order l local approximation of U

$$W = \int_\Omega \frac{W(p(x))}{h(x)^d} \, \mathrm{d}x \tag{22}$$

where

- $p(x)$ local order of approximation
- $W(p)$ computational work for pth order approximation
- $h(x)^{-d}$ local # grid points per volume (density)

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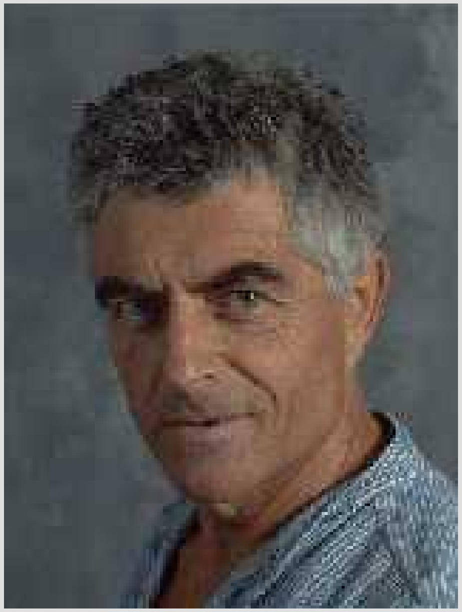
Restrictions on h and p such as uniformity of grids, $p \in \mathbb{N}$, $2\mathbb{N}$.

The Author: Achi Brandt



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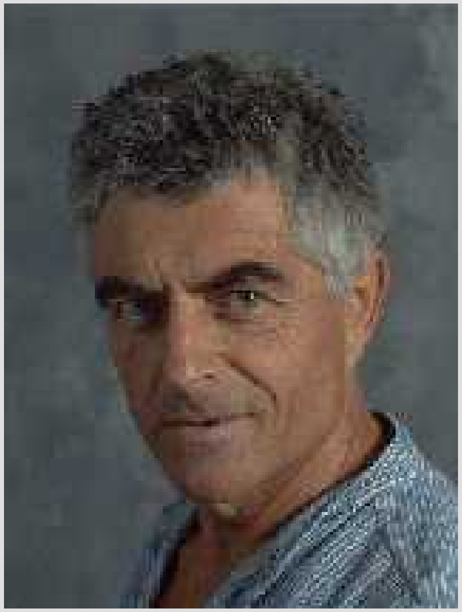

- Born 1938 in Givat-Brener, Israel
- Student at Hebrew University, Jerusalem, 1963 MSc about aspects of random walks
- 1965 PhD from Weizmann Institute of Science, thesis about numerical methods in hydrodynamics and magnetohydrodynamics

- 1972 Associate Professor in Applied Math at the Weizmann Institute, '73-'75 Head of Pure Math Department, '78 - '82 Head of Applied Math Department
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- Worked at The Courant Institute of Mathematical Sciences, New York University, IBM T. J. Watson Research Center, NASA Langley Research Center, Institute for Computational Studies at CSU, Fort Collins, Colorado University at Denver
- Member of Israel Union of Mathematics, SIAM
- Recieved Landau Prize in Mathematics, Rothschild Prize in Mathematics

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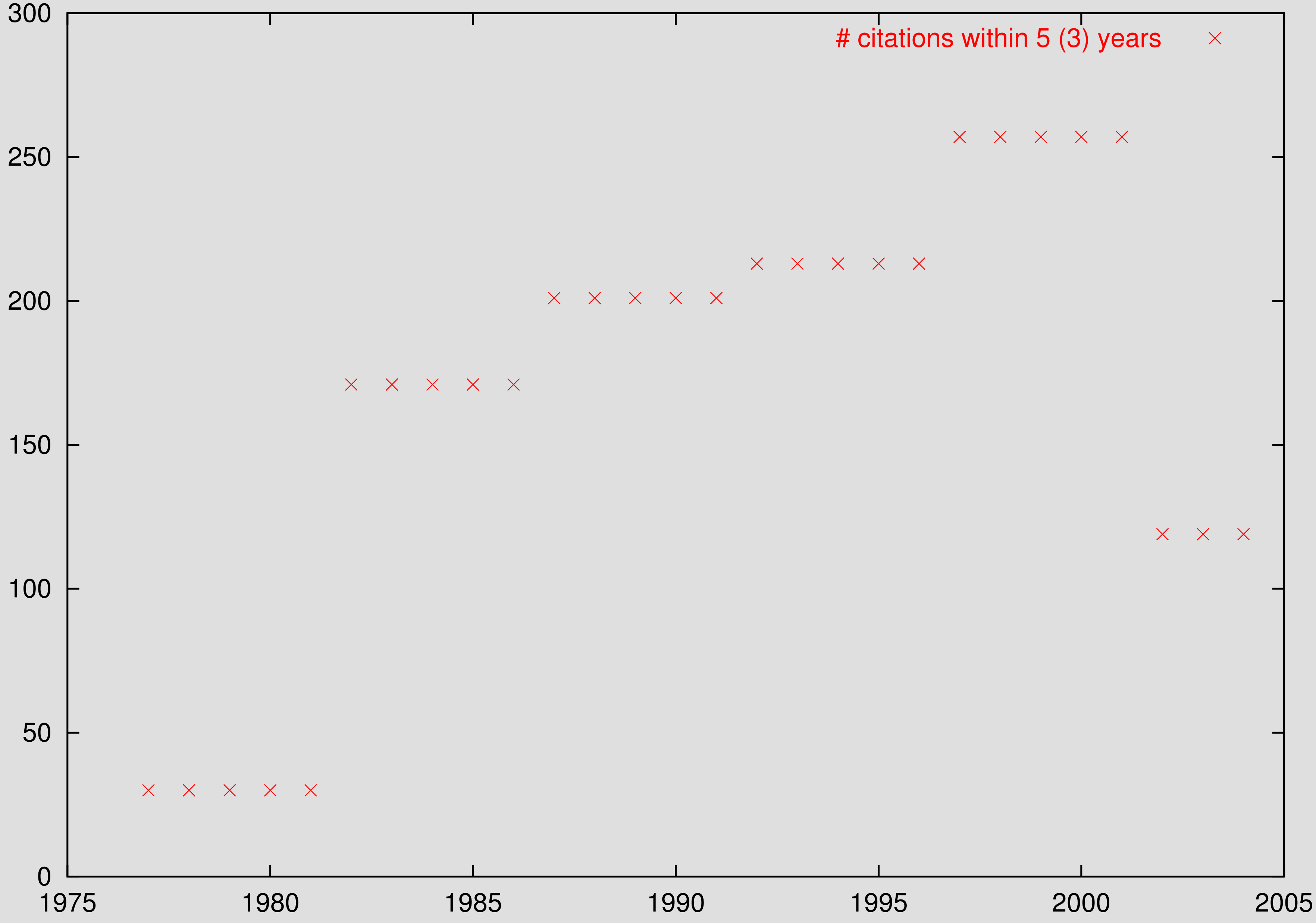

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