Composite Finite Elements for Complicated Geometries

Institute for Numerical Simulation

Lars Ole Schwen, Martin Rumpf

Joint Project with Tobias Preusser, University of Bremen; Stefan Sauter, University of Zurich;
Hans-Joachim Wilke and Uwe Wolfram, University of Ulm

universität bonn

Complicated Domains

Trabecular bones are examples of complicated geometries where we are interested in linearly elastic deformations under compressive load.

Cylindrical specimens are drilled from vertebral bodies and scanned by μ-CT, then the trabecular structure is segmented from the volume data.

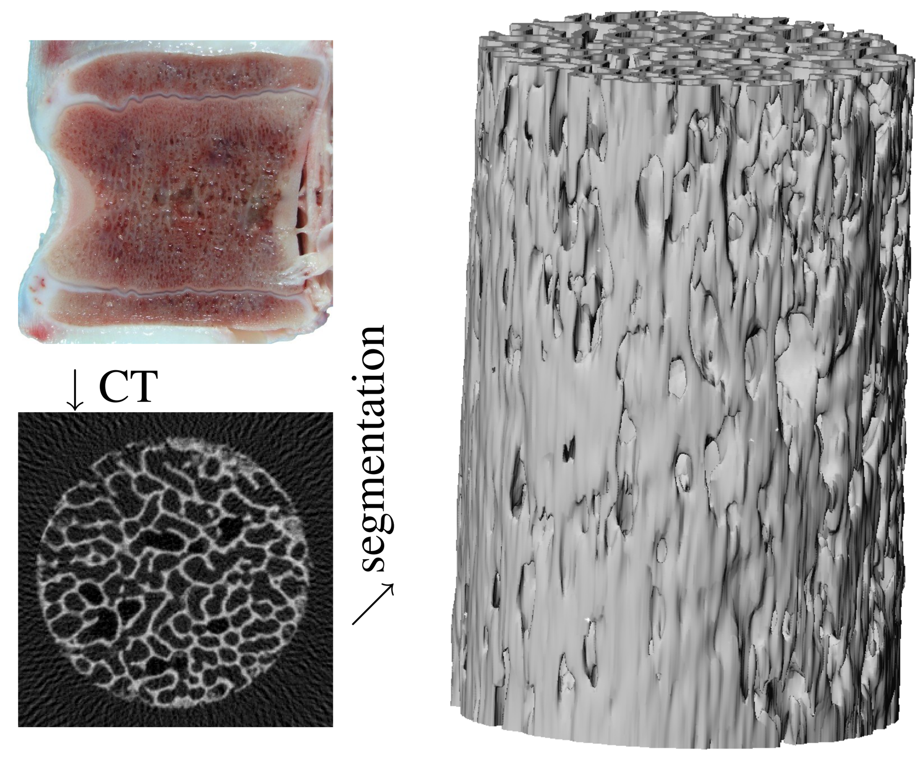

Figure 1: *Example of complicated geometries: photo, CT-Scan, segmented volume*

Classical vs. Composite Finite Elements

Classical Finite Elements:

- meshing necessary
- geometrically flexible grid with simple basis functions
- explicit storage of unstructured grid
- key property: *grid carries complexity*

Composite Finite Elements:

- structured grid with complicated basis functions
- canonical coarse scales will allow multigrid
- key property: *basis functions carry complexity*

Composite Finite Elements

The idea of CFE is most easily explained in 1D: We start with an equidistant grid and piecewise affine-linear basis functions which are set to zero outside the object.

This implies degrees of freedom inside the object (red line) and one layer outside, as shown in Figure 2.

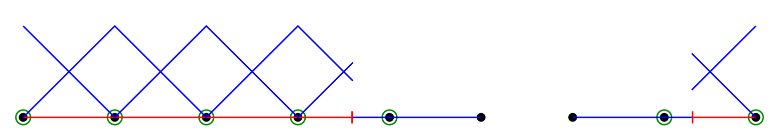

Figure 2: *CFE basis functions in 1D*

In 2D (3D), we start with a uniform quadratic (cubic) grid divided in two triangles (six tetrahedra) each (blue). For reconstructing the domain boundaries (green line), zero-crossings of the trilinear volume data on those vertices need to be determined (red line).

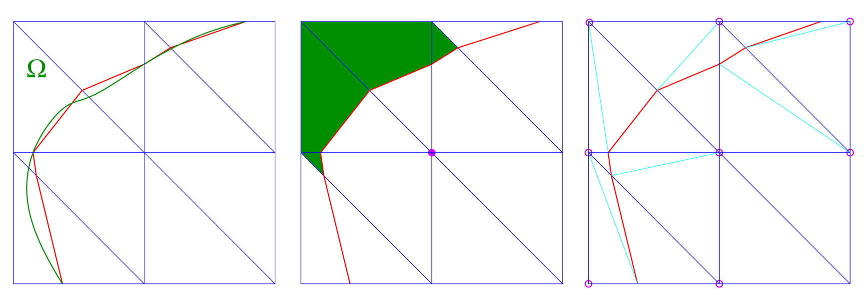

Figure 3: *CFE in 2d: interface recovery, support of one CFE basis function, virtual grid and DOFs*

Basis functions are then set to zero outside the reconstructed object, one such support is shown as the green area.

Internally, the CFE basis functions are composed of basis functions on the so-called virtual grid shown on the right in cyan. Magenta circles show where DOFs are located.

CFE and Multigrid

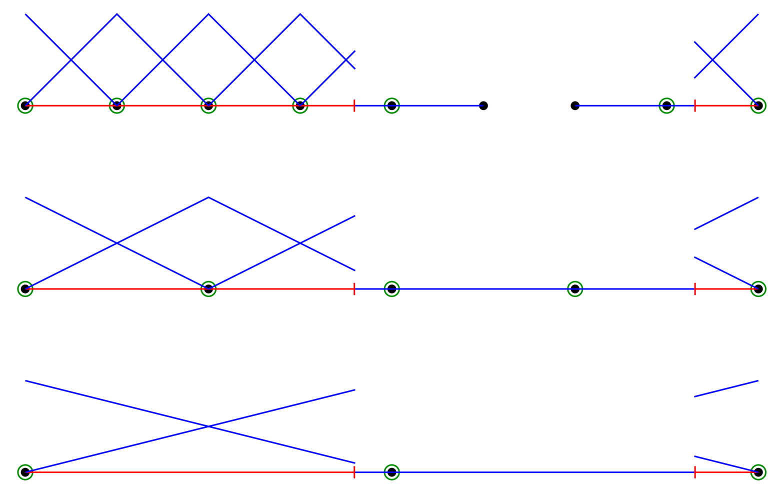

Figure 4: *Canonical grid-based coarsening in 1D*

Note that additional DOFs are introduced on coarse scales (one coarse grid layer outside the object).

However, this purely grid-based strategy is problematic (and leads to poor convergence rates of almost 1) because such coarse grid basis functions may cover physically almost unrelated parts of the object:

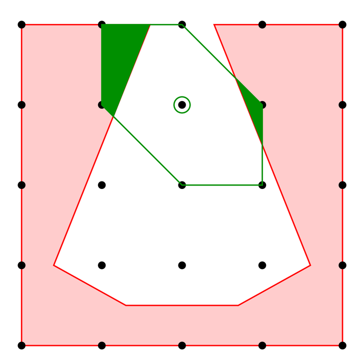

Figure 5: *Coarse-grid basis function in 2D with a two-component support (green) on "horseshoe" domain (red)*

A more sophisticated coarsening method avoiding this effect is currently being developed.

Applications

Heat Conduction

As a scalar PDE problem, we consider heat diffusion with a source term f (constant in time), zero Neumann boundary conditions and implicite Euler timesteps.

For solving the linear systems, we use a multigrid solver with $V(2,2)$-cycles and Gauß-Seidel smoothing.

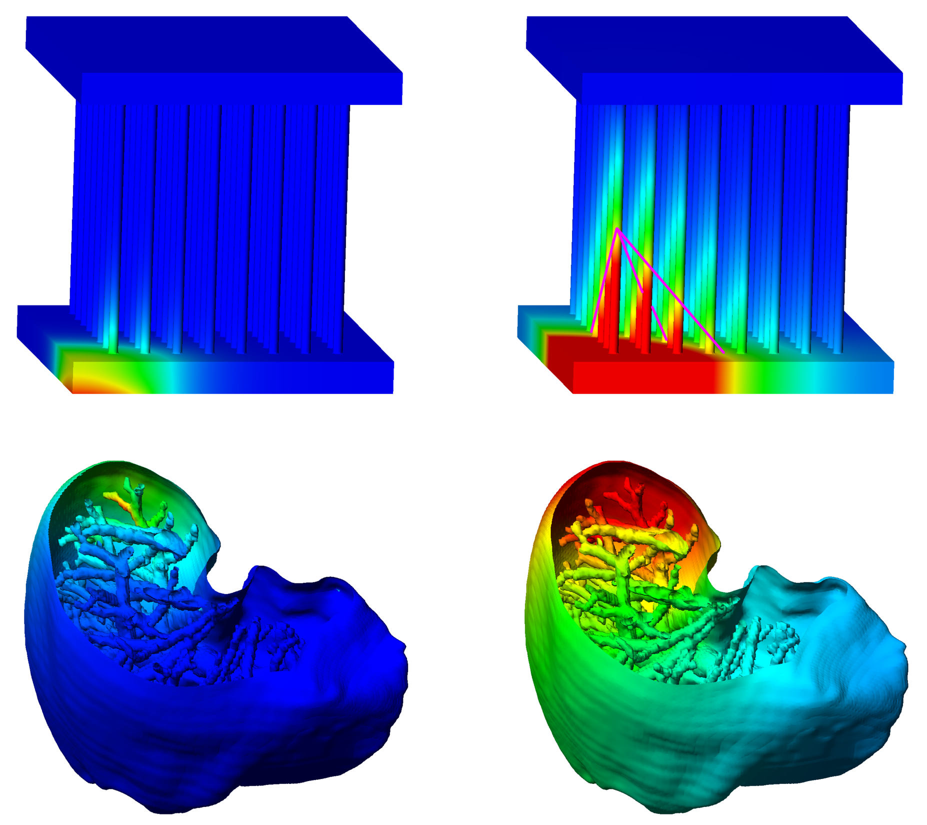

Figure 6: *Two timesteps of a heat diffusion simulation on an artificial geometry and a liver dataset. Macroscopic isothermal lines are shown in magenta.*

Linear Elasticity

As a vector valued problem, we consider linear Lamé-Navier elasticity without volume forces and Dirichlet boundary conditions (top/bottom).

Again, a multigrid solver with $V(2,2)$ cycles and a block variant of Gauß-Seidel smoothing is used.

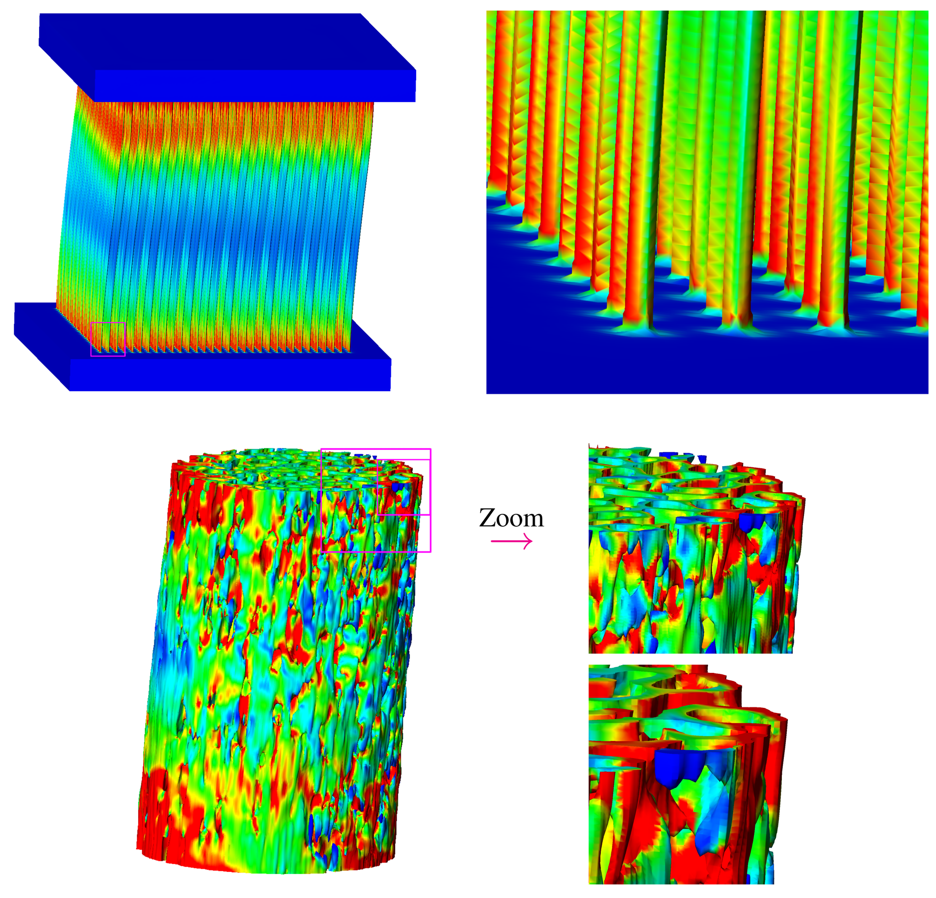

Figure 7: *Deformed artificial geometry and bone specimen*

Outlook

CFE for Jumping Coefficients

For two-phase materials with discontinuous material coefficients across the interface, CFE basis functions with kinks (continuous, but not continuously differentiable across the interface) will be used.

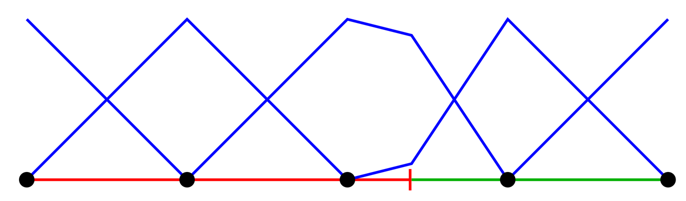

Figure 8: *Basis functions with kinks for materials with jumping coefficients in 1D*

In contrast to 1D, kinks can only be represented approximately in 2D and 3D.

A multigrid solver will also need an appropriate coarsening strategy to avoid the "horseshoe problem" (cf. Fig. 5).

Experimental Validation

Compressive load applied to Aluminum foam and trabecular bone specimens:

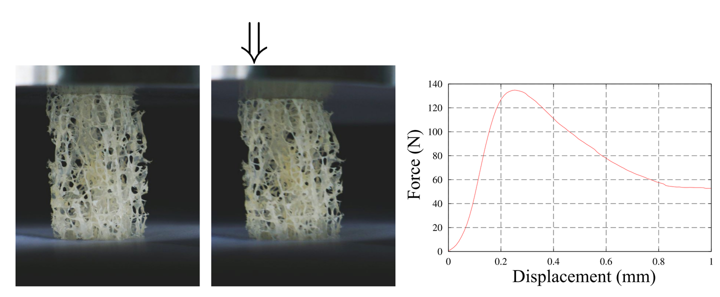

Figure 9: *Compression Experiments for Validation*

References

- W. Hackbusch and S. Sauter: Composite finite elements for the approximation of PDEs on domains with complicated micro-structures, *Numer. Math.*, 75:447–472, 1997.
- S. Sauter and R. Warnke: Composite finite elements for elliptic boundary value problems with discontinuous coefficients, *Computing* 77:29–55, 2006.
- F. Liehr, Ein effizienter Löser für elastische Mikrostrukturen, Diplom thesis, University of Duisburg, 2004.
- F. Liehr, T. Preusser, M. Rumpf, S. Sauter and L. O. Schwen: Composite Finite Elements for 3D Image Based Computing, submitted to *Computing and Visualization in Science*, 2007.
- T. Preusser, M. Rumpf, S. Sauter, L. O. Schwen et al, Three-dimensional composite finite elements for scalar problems with jumping coefficients, 2007, in preparation.